

Delta Journal of Science

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<https://djs.journals.ekb.eg/>



Research Article

MATHEMATICS

Numerical Simulation for the Two-Dimensional Partial Differential Equations

A. A. Soliman¹, K. R. Raslan², Khalid K. Ali², Ahmed S. Awad^{*1}

¹Department of Mathematics, Faculty of Science, Al- Arish University, Al- Arish, 45111, Egypt.

²Department of Mathematics, Faculty of Science, Azhar University, Nasr-City, Cairo, Egypt

***Corresponding author:** Ahmed Awad

e-mail: sobehs50@gmail.com

soliman_99@yahoo.com, kamal raslan@yahoo.com

Received: 26/8/2025

Accepted: 12/10/2025

KEY WORDS

Heat equation,
Wave equation,
Finite
Differences
Methods
(FDM), Non-
Standard Finite
Differences
Method
(NSFDM),
stability
analysis

ABSTRACT

The present paper is devoted to the development of a new schemes for solving the two-dimensional heat equation and two-dimensional wave equation by the finite difference method (FDM) and non-standard finite difference method (NSFDM). The stability analysis for the two schemes is discussed and provides conditional stability conditions. The local truncation errors of both schemes are calculated, so confirm their consistency and convergence of the schemes. Numerical results are obtained for two different test cases for each equation and are compared with exact solutions and other numerical results revealing absolute errors and generating convergence tables. The graphical results show the spatial-temporal behavior of solutions, emphasizing the effectiveness of the presented schemes. When we compare the FDM with the NSFDM, the NSFDM technique outperforms the latter in terms of numerical artifact mitigation. This paper presents a comprehensive methodology for solving multidimensional PDEs while maintaining theoretical rigor (stability, error analysis) and practical validation (numerical benchmarks), with applications in computational physics and engineering.

Introduction

The heat equation is a second-order parabolic PDE that models how temperature (or heat content) evolves over time within a region. In the two-dimensional case, it is commonly formulated as

$$u_t = \alpha(u_{xx} + u_{yy}) \quad (1)$$

where u represents the temperature, t is time, and α is thermal diffusivity (Fourier, 1878). The heat equation is used extensively to design and analyze systems, as well as to improve the thermal efficiency of buildings and machinery (Carslaw and Jaeger, 1959). Moreover, the importance of studying the heat equation strongly appears in the design of electronic components such as microchips; it is essential for predicting and managing heat distribution. Effective thermal management prevents overheating, which can lead to component failure (Ozisik, 1993). Additionally, it is used in medical treatments, such as hyperthermia therapy for cancer, where meticulous regulation of temperature is essential to eradicate tumor cells while preserving adjacent healthy tissue (Incropera and Dewitt, 2002). Recent studies have expanded the applications and deepened the understanding of the heat equation. Advanced numerical methods and computational techniques have improved the accuracy of heat transfer simulations in complex systems (Kumar, 2018). Additionally, the research into nonlinear heat equations has opened new avenues for modeling more complex thermal phenomena (Mainardi, 2010). Innovations in materials science have led to the development of materials with tailored thermal properties, enhancing the efficiency of heat transfer

in various applications (Cahill, 2014). Many numerical methods are used for solving the nonlinear PDEs (Soliman et al., 2007a; 2007b; 2008; 2009a; 2009b; 2009c; 2012a; 2012b; 2021). This equation provides a framework for analyzing temperature distribution across surfaces, such as metal plates or electronic components. Understanding the two-dimensional heat equation is crucial for optimizing thermal management strategies, predicting heat transfer in complex materials, and simulating environmental temperature changes (Gonçalves and Gomes, 2022; AZoM, 2019). The wave equation is a second-order linear PDE that models the propagation of different wave types, including acoustic waves, electromagnetic waves, and water waves, through physical media. Mathematically, it is expressed as

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad (2)$$

where u denotes the wave function, t is the time, and c denotes the constant wave propagation velocity. Studying the wave equation is crucial in multiple fields such as acoustics, electromagnetism and fluid dynamics. This equation helps understand how waves travel through different media, which has practical applications in designing acoustic devices, electromagnetic systems and aerodynamics (Wikipedia). There are many types of wave equations, such as the linear wave equation, the nonlinear wave equation, the multi-dimensional wave equation and the damped wave equation (Komatitsch et al., 1996). The applications of the wave equation are vast and varied, spanning across multiple scientific domains. Electromagnetic waves

are described by Maxwell's equations, which are pivotal in understanding the propagation of electromagnetic waves through different media. Acoustic waves are essential in the design of acoustic systems, including microphones and speakers, facilitating advancements in sound technology. Lastly, seismic waves are crucial for studying wave propagation through the Earth's interior, aiding in the analysis and understanding of earthquakes. These applications highlight the fundamental role of wave equations in scientific research and particle technology (Li et al., 2017; Jaaskelainen, 2010; Chen, 2010). In recent years numerous studies have been conducted on the wave equation to enhance our understanding of superwaves and nuclear waves. For example, the wave equation has been utilized in the study of superwaves to improve wired and wireless communications. Additionally, it has been applied in the study of nuclear waves to understand particle interactions and develop safe nuclear energy technologies (Bezerra, 2010). This paper is organized as follows: First, we present the two-dimensional heat and wave equations. Next, we introduce both finite difference and non-standard finite difference schemes for these equations. Our discussion includes stability analysis for both approaches when applied to the two-dimensional heat and wave equations, along with an evaluation of their local truncation errors. The numerical results, including a comparison with the exact solutions and relevant figures, are presented. We develop conditionally stable schemes; one is based on the finite difference method, and the other on the non-standard approach. The stability analysis for the schemes is detailed, demonstrating their conditional stability.

Numerical results are presented and analyzed, followed by concluding remarks.

The two-dimensional heat equation

The two-dimensional heat equation belongs to the class of parabolic partial differential equations that models the diffusion of heat in isotropic materials. It is solved using analytical methods, for example, the separation of variables or using numerical methods, including finite difference, finite element, and spectral approaches. The two-dimensional heat equation has various applications in science and engineering such as thermal analysis of solids, geophysics and environmental science, biomedical engineering and material science. The two-dimensional heat equation takes the form

$$u_t(x, y, t) = \alpha[u_{xx}(x, y, t) + u_{yy}(x, y, t)],$$

where $x, y \in [l, m]$,

The boundary conditions

$$u(0, y, t) = u(1, y, t) = 0, \quad \forall t > 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad \forall t > 0,$$

with initial conditions

$$u(x, y, 0) = f(x, y),$$

The analytical solution is: (Grema et al., 2022)

$$u(x, y, t) = e^{-2\alpha\pi^2 t} \sin(\pi x) \sin(\pi y).$$

Finite Difference Scheme

To numerically solve the two-dimensional heat equation using the finite difference method, we first derive discrete approximations for the partial derivatives.

$$(u_{ij}^n)_t = \frac{1}{k} [u_{ij}^{n+1} - u_{ij}^n] + O(k) \quad (3)$$

$$(u_{ij}^n)_{xx} = \frac{1}{h^2} [u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n] + O(h)^2 \quad (4)$$

$$(u_{ij}^n)_{yy} = \frac{1}{h^2} [u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n] + O(h)^2 \quad (5)$$

where $\Delta t = k$ and $\Delta x = \Delta y = h$.

The finite difference scheme of the Eq. (1) can be written as follows:

$$(U_{ij}^n)_t - \alpha \left[(U_{ij}^n)_{xx} + (U_{ij}^n)_{yy} \right] = 0 \quad (6)$$

By using the difference approximations provided in relations (3), (4) and (5) into Eq. (6), then we get:

$$\frac{1}{k} [U_{ij}^{n+1} - U_{ij}^n] = \frac{\alpha}{h^2} [U_{i+1,j}^n - 2U_{ij}^n + U_{i-1,j}^n] + \frac{\alpha}{h^2} [U_{i,j+1}^n - 2U_{ij}^n + U_{i,j-1}^n] \quad (7)$$

which can be written further as:

$$U_{ij}^{n+1} = U_{ij}^n + r \left([U_{i+1,j}^n - 2U_{ij}^n + U_{i-1,j}^n] + [U_{i,j+1}^n - 2U_{ij}^n + U_{i,j-1}^n] \right) \quad (8)$$

Then,

$$U_{ij}^{n+1} = rU_{i-1,j}^n + rU_{i+1,j}^n + (1-4r)U_{ij}^n + rU_{i,j-1}^n + rU_{i,j+1}^n \quad (9)$$

where $r = \frac{\alpha k}{h^2}$, $x_i = ih$, $y_j = jh$, and $t_n = nk$, with $i, j = 0, 1, \dots$ and $n = 0, 1, \dots$. Here, the superscript n indicates the quantity corresponding to a specific time level t_n and the subscripts i, j represent the value associated with the specific spatial mesh points (x_i, y_j) . Let u_{ij}^n represent the analytical solution at the grid point (x_i, y_j, t_n) , while U_{ij}^n denotes the corresponding approximate numerical value at the same grid point. This scheme enables the formulation of an algebraic system from which the numerical solution is then derived.

Non-standard finite difference scheme

To numerically solve the two-dimensional heat equation using the nonstandard finite difference method, we first derive discrete approximations for the partial derivatives.

$$(U_{ij}^n)_t = \frac{1}{\phi(k)} [U_{ij}^{n+1} - U_{ij}^n] + O(k) \quad (10)$$

$$(U_{ij}^n)_{xx} = \frac{1}{\beta(h)^2} [U_{i+1,j}^n - 2U_{ij}^n + U_{i-1,j}^n] + O(h)^2 \quad (11)$$

$$(U_{ij}^n)_{yy} = \frac{1}{\gamma(h)^2} [U_{i,j+1}^n - 2U_{ij}^n + U_{i,j-1}^n] + O(h)^2 \quad (12)$$

where $\Delta t = k$ and $\Delta x = \Delta y = h$.

Then, we can write the nonstandard finite difference scheme of the Eq. (1) as follows:

$$\frac{1}{\phi(k)} [U_{ij}^{n+1} - U_{ij}^n] = \frac{\alpha}{\beta(h)^2} [U_{i+1,j}^n - 2U_{ij}^n + U_{i-1,j}^n] + \frac{\alpha}{\gamma(h)^2} [U_{i,j+1}^n - 2U_{ij}^n + U_{i,j-1}^n] \quad (13)$$

The nonstandard finite difference scheme is

$$U_{ij}^{n+1} = rU_{i-1,j}^n + rU_{i+1,j}^n + (1-4r)U_{ij}^n + rU_{i,j-1}^n + rU_{i,j+1}^n \quad (14)$$

where $r = \frac{\alpha \phi(k)}{\beta(h)^2} = \frac{\alpha \phi(k)}{\gamma(h)^2}$, $\phi(k) = \sin(k)$, $\beta(h) = \sin(h)$, $\gamma(h) = \sin(h)$, $x_i = ih$,

$y_j = jh$, and $t_n = nk$, with $i, j = 0, 1, \dots$ and $n = 0, 1, \dots$. Here, the superscript n indicates the quantity corresponding to a specific time level t_n and the subscripts i, j represent the value associated with the specific spatial mesh points (x_i, y_j) . Let u_{ij}^n represent the analytical solution at the grid point (x_i, y_j, t_n) , while U_{ij}^n denotes the corresponding approximate numerical value at the same grid point. This scheme enables the formulation of an algebraic system from which the numerical solution is then derived.

Stability analysis of finite difference scheme

Lemma (1): The scheme (9) exhibits conditional stability.

Proof. We employ Von-Neumann stability analysis to examine the scheme's stability through the following approach

$$U_{ij}^n = \xi^n e^{Ih(ik_x + jk_y)}, \quad I = \sqrt{-1} \quad (15)$$

where k_x, k_y are the mode numbers and h represents the element size. By substituting Eq. (15) in Eq. (9) leads to

$$\xi^{n+1} = g\xi^n \quad (16)$$

where g denotes the growth factor, from Eqs. (16) and (9) we obtain

$$g\xi^n = \xi^n \{r(e^{-Ihk_x} + e^{Ihk_x}) + r(e^{-Ihk_y} + e^{Ihk_y}) + (1 - 4r)\} \quad (17)$$

$$g\xi^n = \xi^n \{1 - 2r(1 - \cos k_x h) - 2r(1 - \cos k_y h)\} \quad (18)$$

$$g = \left[1 - 4r \sin^2\left(\frac{k_x h}{2}\right) - 4r \sin^2\left(\frac{k_y h}{2}\right)\right] \quad (19)$$

Now, we can prove that

$$\left|1 - 4r \sin^2\left(\frac{k_x h}{2}\right) - 4r \sin^2\left(\frac{k_y h}{2}\right)\right| \leq 1 \quad (20)$$

$$-1 \leq 1 - 8r \leq 1 \quad (21)$$

Therefore

$$0 \leq r \leq \frac{1}{4} \quad (22)$$

Where $r = \frac{\alpha k}{h^2}$

Thus, the scheme is conditionally stable.

Stability analysis of non-standard finite difference scheme

Lemma (2): The scheme (14) exhibits conditional stability.

Proof. We employ Von-Neumann stability analysis to examine the scheme's stability through the following approach

$$U_{ij}^n = \xi^n e^{I(ik_x \beta(\Delta x) + jk_y \gamma(\Delta y))}, \quad I = \sqrt{-1} \quad (23)$$

Where k_x, k_y are the mode numbers and $\Delta x, \Delta y$ represent the element sizes.

Now, substituting Eq. (23) into Eq. (14) leads to

$$\xi^{n+1} = g\xi^n \quad (24)$$

where g denotes the growth factor, and from Eqs. (24) and (14) we get

$$g\xi^n = \xi^n \{r_1(e^{-I\beta(\Delta x)k_x} + e^{I\beta(\Delta x)k_x}) + r_2(e^{-I\gamma(\Delta y)k_y} + e^{I\gamma(\Delta y)k_y}) + (1 - 2r_1 - 2r_2)\} \quad (25)$$

$$g\xi^n = \xi^n \{1 - 2r_1(1 - \cos k_x \beta(\Delta x)) - 2r_2(1 - \cos k_y \gamma(\Delta y))\} \quad (26)$$

$$g = \left[1 - 4r_1 \sin^2\left(\frac{k_x \beta(\Delta x)}{2}\right) - 4r_2 \sin^2\left(\frac{k_y \gamma(\Delta y)}{2}\right)\right] \quad (27)$$

Now, we can prove that

$$\left|1 - 4r_1 \sin^2\left(\frac{k_x \beta(\Delta x)}{2}\right) - 4r_2 \sin^2\left(\frac{k_y \gamma(\Delta y)}{2}\right)\right| \leq 1 \quad (28)$$

$$0 \leq 4r_1 \sin^2\left(\frac{k_x \beta(\Delta x)}{2}\right) + 4r_2 \sin^2\left(\frac{k_y \gamma(\Delta y)}{2}\right) \leq 2 \quad (29)$$

$$0 \leq r_1 + r_2 \leq \frac{1}{2} \quad (30)$$

Hence,

$$\frac{\alpha \phi(\Delta t)}{\beta(\Delta x)^2} + \frac{\alpha \phi(\Delta t)}{\gamma(\Delta y)^2} \leq \frac{1}{2} \quad (31)$$

$$\text{Where } r_1 = \frac{\alpha \phi(\Delta t)}{\beta(\Delta x)^2}, \quad r_2 = \frac{\alpha \phi(\Delta t)}{\gamma(\Delta y)^2}$$

Therefore, the scheme is conditionally stable.

Local truncation error of finite difference scheme

Lemma (3): The local truncation error $\tau_{i,j}^n$ of the scheme (9) is $O(\Delta t + (\Delta x)^2 + (\Delta y)^2)$.

Proof. To estimate the truncation error of scheme (9), we perform a Taylor expansion. The local truncation error is given by:

$$\tau_{i,j}^n = \frac{1}{\Delta t} [u_{i,j}^{n+1} - u_{i,j}^n] - \frac{\alpha}{(\Delta x)^2} [u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n] - \frac{\alpha}{(\Delta y)^2} [u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n] \quad (32)$$

using Taylor's series expansion, we get

$$\frac{1}{\Delta t} (u_{i,j}^{n+1} - u_{i,j}^n) = \left(\frac{\partial u}{\partial t} \right)_{i,j}^n + \frac{\Delta t}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n + \dots \quad (33)$$

$$\frac{1}{(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) = \left(\frac{\partial^2 u}{\partial x^2} \right)_{i,j}^n + \frac{(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n + \dots \quad (34)$$

$$\frac{1}{(\Delta y)^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) = \left(\frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \frac{(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n + \dots \quad (35)$$

substitution from (33),(34)and(35) into (32)

$$\tau_{i,j}^n = \left(\frac{\partial u}{\partial t} \right)_{i,j}^n - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \frac{\Delta t}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - \frac{\alpha(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \frac{\alpha(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n \quad (36)$$

But is the solution of differential Eq. (1), so

$$\left(\frac{\partial u}{\partial t} \right)_{i,j}^n - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n = 0 \quad (32)$$

Therefore, the foremost term in the local truncation error is given by

$$\frac{\Delta t}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - \frac{\alpha(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \frac{\alpha(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n$$

Then, the local truncation error is

$$\tau_{i,j}^n = O(\Delta t + (\Delta x)^2 + (\Delta y)^2) \quad (37)$$

Local truncation error of non-Standard finite difference scheme

Lemma (4): The local truncation error $\tau_{i,j}^n$ of the scheme (14) is $O(\phi(\Delta t) + \beta(\Delta x)^2 + \gamma(\Delta y)^2)$.

Proof. To estimate the truncation error of scheme (14), we perform a Taylor expansion.

The local truncation error is given by:

$$\tau_{i,j}^n = \frac{1}{\phi(\Delta t)} [u_{i,j}^{n+1} - u_{i,j}^n] - \frac{\alpha}{\beta(\Delta x)^2} [u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n] - \frac{\alpha}{\gamma(\Delta y)^2} [u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n] \quad (38)$$

using Taylor's series expansion, we get

$$\frac{1}{\phi(\Delta t)} (u_{i,j}^{n+1} - u_{i,j}^n) = \left(\frac{\partial u}{\partial t} \right)_{i,j}^n + \frac{\phi(\Delta t)}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n + \dots \quad (39)$$

$$\frac{1}{\beta(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) = \left(\frac{\partial^2 u}{\partial x^2} \right)_{i,j}^n + \frac{\beta(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n + \dots \quad (40)$$

$$\frac{1}{\gamma(\Delta y)^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) = \left(\frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \frac{\gamma(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n + \dots \quad (41)$$

substitution from (39), (40) and (41) into (38)

$$\tau_{i,j}^n = \left(\frac{\partial u}{\partial t} \right)_{i,j}^n - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \frac{\phi(\Delta t)}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - \frac{\alpha \beta(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \frac{\alpha \gamma(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n \quad (42)$$

But is the solution of differential Eq. (1),
so

$$\left(\frac{\partial u}{\partial t}\right)_{i,j}^n - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{i,j}^n = 0$$

Then, the local truncation error is

$$\tau_{i,j}^n = O(\phi(\Delta t) + \beta(\Delta x)^2 + \gamma(\Delta y)^2) \quad (43)$$

Numerical Results

This section presents the numerical results for the two-dimensional heat equation using Test Problem (1). Both the FDM and NSFDM were employed to solve the partial differential equation, verifying the schemes' accuracy and computational efficiency. The absolute error, defined as the difference between the numerical and exact solutions, was used as a measure of accuracy. The results are summarized and presented in Tables [1–2].

Test Problem (1):

The two-dimensional heat equation is expressed as

$$u_t(x, y, t) = \alpha[u_{xx}(x, y, t) + u_{yy}(x, y, t)],$$

Where $x, y \in [0, 1]$

The boundary conditions

$$u(0, y, t) = u(1, y, t) = 0, \quad \forall t > 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad \forall t > 0,$$

with initial conditions

$$u(x, y, 0) = f(x, y),$$

The analytical solution is: (Grema et al., 2022)

$$u(x, y, t) = e^{-2\alpha\pi^2 t} \sin(\pi x) \sin(\pi y).$$

Let use steps sizes $h = 0.05$, $k = 0.0001$, $M = 5000$, $N = 20$ and $\alpha = 1$. Now, we use the FDM and the NSFDM to find numerical solutions for this example.

Table (1): Analytical and numerical results at $y=0.5$ using FDM

x_i	Analytical solution	FDM	Absolute error
0	0	0	0
0.05	8.09129×10^{-6}	8.17721×10^{-6}	8.59229×10^{-8}
0.1	0.0000159833	0.0000161531	1.6973×10^{-7}
0.15	0.0000234818	0.0000237312	2.49358×10^{-7}
0.2	0.0000304021	0.000030725	3.22846×10^{-7}
0.25	0.0000365738	0.0000369622	3.88384×10^{-7}
0.3	0.0000418449	0.0000422893	4.44359×10^{-7}
0.35	0.0000460857	0.0000465751	4.89392×10^{-7}
0.4	0.0000491917	0.000049714	5.22375×10^{-7}
0.45	0.0000510864	0.0000516289	5.42496×10^{-7}
0.5	0.0000517232	0.0000522724	5.49258×10^{-7}
0.55	0.0000510864	0.0000516289	5.42496×10^{-7}
0.6	0.0000491917	0.000049714	5.22375×10^{-7}
0.65	0.0000460857	0.0000465751	4.89392×10^{-7}
0.7	0.0000418449	0.0000422893	4.44359×10^{-7}
0.75	0.0000365738	0.0000369622	3.88384×10^{-7}
0.8	0.0000304021	0.000030725	3.22846×10^{-7}
0.85	0.0000234818	0.0000237312	2.49358×10^{-7}
0.9	0.0000159833	0.0000161531	1.6973×10^{-7}
0.95	8.09129×10^{-6}	8.17721×10^{-6}	8.59229×10^{-8}
1	6.33426×10^{-21}	0	6.33426×10^{-21}

Table (1) gives the approximate solution for this example at $y = 0.5$ and its comparison to the analytical solution. Includes the absolute error. The maximum value of error for this example in the interval $[0, 1]$ is

$$l_{\infty} = 5.49258 \times 10^{-7}$$

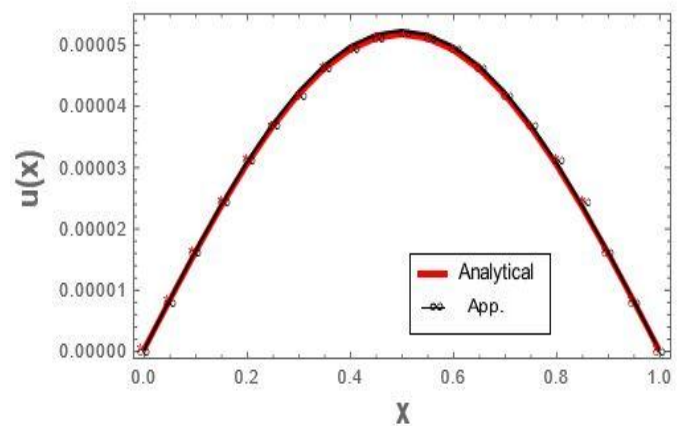


Fig. (1): The numerical outcomes and the analytical solutions by FDM at $y=0.5$

Table (2): Analytical and numerical results at $y=0.5$ using NSFDM

x_i	Analytical solution	NS-FDM	Absolute error
0	0	0	0
0.05	8.09129×10^{-6}	8.11021×10^{-6}	1.89162×10^{-8}
0.1	0.0000159833	0.0000160207	3.73666×10^{-8}
0.15	0.0000234818	0.0000235367	5.48969×10^{-8}
0.2	0.0000304021	0.0000304732	7.10755×10^{-8}
0.25	0.0000365738	0.0000366593	8.55039×10^{-8}
0.3	0.0000418449	0.0000419428	9.7827×10^{-8}
0.35	0.0000460857	0.0000461934	1.07741×10^{-7}
0.4	0.0000491917	0.0000493067	1.15003×10^{-7}
0.45	0.0000510864	0.0000512058	1.19432×10^{-7}
0.5	0.0000517232	0.0000518441	1.20921×10^{-7}
0.55	0.0000510864	0.0000512058	1.19432×10^{-7}
0.6	0.0000491917	0.0000493067	1.15003×10^{-7}
0.65	0.0000460857	0.0000461934	1.07741×10^{-7}
0.7	0.0000418449	0.0000419428	9.7827×10^{-8}
0.75	0.0000365738	0.0000366593	8.55039×10^{-8}
0.8	0.0000304021	0.0000304732	7.10755×10^{-8}
0.85	0.0000234818	0.0000235367	5.48969×10^{-8}
0.9	0.0000159833	0.0000160207	3.73666×10^{-8}
0.95	8.09129×10^{-6}	8.11021×10^{-6}	1.89162×10^{-8}
1	6.33426×10^{-21}	0	6.33426×10^{-21}

Table (2) gives the approximate solution for this example at $y = 0.5$ and its comparison to the analytical solution. Includes the absolute error. The maximum value of error for this example in the interval $[0, 1]$ is

$$l_{\infty} = 1.20921 \times 10^{-7}$$

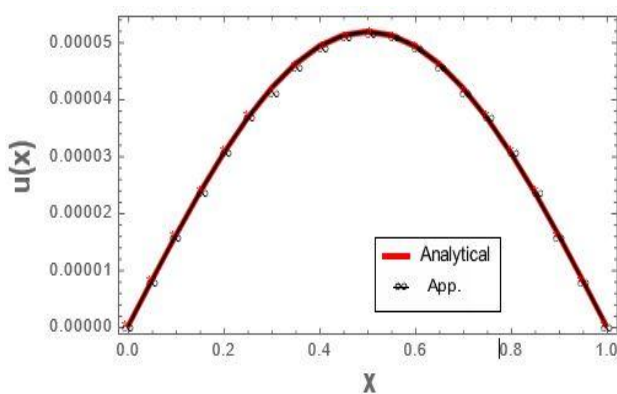


Fig. (2): The numerical outcomes and the analytical solutions by NSFDM at $y=0.5$

In Figures [1, 2], for $x \in [0, 1]$ at $y=0.5$, the numerical results are represented by a black curve, while analytical solutions are shown as a red curve. We conclude that the analytical and numerical solutions obtained by NSFDM are very close, which gives more accuracy to the solution.

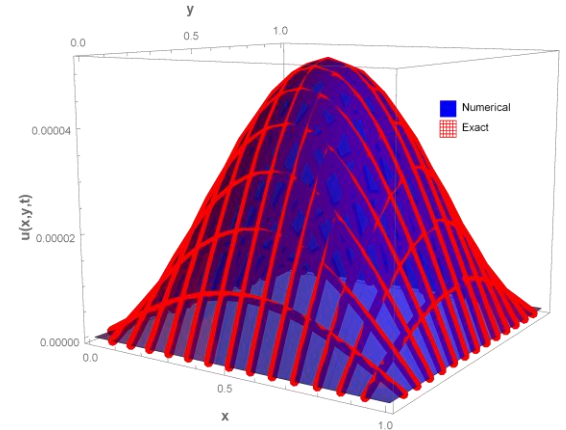


Fig. (3): The numerical outcomes and the analytical solutions by FDM

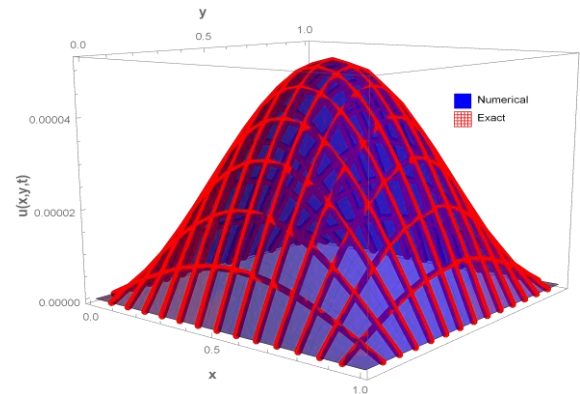


Fig. (4): The numerical outcomes and the analytical solutions by NSFDM

In Figures [3, 4], within the interval $[0, 1]$, the numerical results are represented by a blue solid surface, while the analytical solutions are shown as a transparent surface with red mesh. We conclude that the analytical and numerical solutions obtained by NSFDM are very close, which gives more accuracy to the solution.

The two-dimensional Wave equation

The two-dimensional wave equation is a hyperbolic PDE, describing the propagation of waves in elastic media, such as sound waves, water waves, or vibrations in a membrane. It is solved using analytical methods for example, the separation of variables or using numerical methods, including finite difference, finite element, and spectral approaches. The two-dimensional wave equation has many practical applications in physics, engineering, and other fields such as acoustics and sound propagation, vibrations in membranes and plates, electromagnetic wave propagation, computer graphics and animation. The two-dimensional wave equation takes the form:

$$u_{tt}(x, y, t) = c^2 [u_{xx}(x, y, t) + u_{yy}(x, y, t)]$$

Where $x, y \in [l, m]$,

The boundary conditions

$$u(0, y, t) = u(1, y, t) = 0, \quad \forall t > 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad \forall t > 0,$$

with initial conditions

$$u(x, y, 0) = f(x, y),$$

$$u_t(x, y, 0) = g(x, y),$$

The analytical solution is: (Daileda, 2014)

$$u(x, y, t) = \cos(\sqrt{2\pi}ct) \sin(\pi x) \sin(\pi y).$$

Finite difference scheme

To numerically solve the two-dimensional wave equation using the finite difference method, we first derive discrete approximations for the partial derivatives.

$$(u_{ij}^n)_{tt} = \frac{1}{k^2} [u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}] + o(k)^2 \quad (44)$$

$$(u_{ij}^n)_{xx} = \frac{1}{h^2} [u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n] + o(h)^2 \quad (45)$$

$$(u_{ij}^n)_{yy} = \frac{1}{h^2} [u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n] + o(h)^2 \quad (46)$$

where $\Delta t = k$ and $\Delta x = \Delta y = h$

The finite difference scheme of the Eq. (2) can be written as follows:

$$(u_{ij}^n)_{tt} - c^2 [(u_{ij}^n)_{xx} + (u_{ij}^n)_{yy}] = 0 \quad (47)$$

By using the difference approximations provided in relations (44), (45) and (46) into Eq. (47), then we get:

$$\begin{aligned} \frac{1}{k^2} [u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}] = \\ \frac{c^2}{h^2} [u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n] + \\ \frac{c^2}{h^2} [u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n] \end{aligned} \quad (48)$$

which can be written further as:

$$\begin{aligned} u_{ij}^{n+1} = r^2 u_{i-1,j}^n + r^2 u_{i,j-1}^n + 2(1 - \\ 2r^2) u_{ij}^n + r^2 u_{i+1,j}^n + r^2 u_{i,j+1}^n - u_{ij}^{n-1} \end{aligned} \quad (49)$$

where $r = \frac{ck}{h}$, $x_i = ih$, $y_j = jh$, and $t_n = nk$, with $i, j = 0, 1, \dots$ and $n = 0, 1, \dots$. Here, the superscript n indicates the quantity corresponding to a specific time level t_n and the subscripts i, j represent the value associated with the specific spatial mesh points (x_i, y_j) . Let u_{ij}^n represent the analytical solution at the grid point (x_i, y_j, t_n) , while U_{ij}^n denotes the corresponding approximate numerical value at the same grid point. This scheme enables the formulation of an algebraic system from which the numerical solution is then derived.

Non-standard finite difference scheme

To numerically solve the two-dimensional wave equation using the nonstandard finite difference method, we first derive discrete approximations for the partial derivatives.

$$(u_{ij}^n)_{tt} = \frac{1}{\phi(k)^2} [u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}] + o(k)^2 \quad (50)$$

$$(U_{i,j}^n)_{xx} = \frac{1}{\beta(h)^2} [U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n] + O(h)^2 \quad (51)$$

$$(U_{i,j}^n)_{yy} = \frac{1}{\gamma(h)^2} [U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n] + O(h)^2 \quad (52)$$

where $\Delta t = k$ and $\Delta x = \Delta y = h$. The nonstandard finite difference scheme of the Eq. (2) can be written as follows:

$$(U_{i,j}^n)_{tt} - c^2 [(U_{i,j}^n)_{xx} + (U_{i,j}^n)_{yy}] = 0 \quad (53)$$

By using the difference approximations provided in relations (50), (51) and (52) into Eq. (53), then we get:

$$\begin{aligned} \frac{1}{\phi(k)^2} [U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}] = \\ \frac{c^2}{\beta(h)^2} [U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n] + \\ \frac{c^2}{\gamma(h)^2} [U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n] \end{aligned} \quad (54)$$

which can be written further as:

$$U_{i,j}^{n+1} = r^2 U_{i-1,j}^n + r^2 U_{i,j-1}^n + 2(1 - 2r^2) U_{i,j}^n + r^2 U_{i+1,j}^n + r^2 U_{i,j+1}^n - U_{i,j}^{n-1} \quad (55)$$

where $r = \frac{c \phi(k)}{\beta(h)} = \frac{c \phi(k)}{\gamma(h)}$,
 $\phi(k) = \sin(k)$, $\beta(h) = \sin(h)$, $\gamma(h) = \sin(h)$, $x_i = ih$,

$y_j = jh$, and $t_n = nk$, with $i, j = 0, 1, \dots$ and $n = 0, 1, \dots$. Here, the superscript n indicates the quantity corresponding to a specific time level t_n and the subscripts i, j represent the value associated with the specific spatial mesh points (x_i, y_j) . Let $u_{i,j}^n$ represent the analytical solution at the grid point (x_i, y_j, t_n) , while $U_{i,j}^n$ denotes the corresponding approximate numerical value at the same grid point. This scheme enables the formulation of an algebraic system from which the numerical solution is then derived.

Stability analysis of finite difference scheme

Lemma (5). The scheme (49) exhibits conditional stability.

Proof. We employ Von Neumann stability analysis to examine the scheme's stability through the following approach

$$U_{i,j}^n = \xi^n e^{Ih(ik_x + jk_y)}, \quad I = \sqrt{-1} \quad (56)$$

where k_x, k_y are the mode numbers and h represents the element size.

By substituting Eq. (56) into Eq. (49) leads to

$$\xi^{n+1} = g \xi^n \quad (57)$$

where g denotes the growth factor, from EQs. (57) and (49) we get

$$g^2 - 2g \left[1 - 2r^2 \sin^2 \left(\frac{k_x h}{2} \right) - 2r^2 \sin^2 \left(\frac{k_y h}{2} \right) \right] + 1 = 0 \quad (58)$$

We assume that the roots of equation are g_1 and g_2 . Now, we can prove that

$$\left| 1 - 2r^2 \sin^2 \left(\frac{k_x h}{2} \right) - 2r^2 \sin^2 \left(\frac{k_y h}{2} \right) \right| \leq 1 \quad (59)$$

$$-1 \leq 1 - 2r^2 \sin^2 \left(\frac{k_x h}{2} \right) - 2r^2 \sin^2 \left(\frac{k_y h}{2} \right) \leq 1 \quad (60)$$

That is, $r^2 = \frac{c^2 k^2}{h^2}$ cannot be a negative quantity, that is the inequality (59) leads to $r^2 > 0$, and this is true to satisfy the inequality (60)

$$0 \leq 2r^2 \sin^2 \left(\frac{k_x h}{2} \right) + 2r^2 \sin^2 \left(\frac{k_y h}{2} \right) \leq 2 \quad (61)$$

Therefore

$$\frac{-1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}} \quad (62)$$

Then the equation (58) can be written as

$$g^2 - 2g \cos \zeta + 1 = 0$$

$$g = \cos \zeta \pm \sqrt{\cos^2 \zeta - 1} \quad (63)$$

Where

$$\cos \zeta = 1 - 2r^2 \sin^2 \left(\frac{k_x h}{2} \right) - 2r^2 \sin^2 \left(\frac{k_y h}{2} \right),$$

thus yields $|g_1| = |g_2| = 1$.

Thus, the scheme is conditionally stable.

Stability analysis of non-standard finite difference scheme

Lemma (6): The scheme (55) exhibits conditional stability.

Proof. We employ Von Neumann stability analysis to examine the scheme's stability through the following approach

$$U_{i,j}^n = \xi^n e^{I(ik_x \beta(\Delta x) + jk_y \gamma(\Delta y))}, \quad I = \sqrt{-1} \quad (64)$$

Where k_x, k_y are the mode numbers and $\Delta x, \Delta y$ represent the element sizes. Now, substituting Eq. (64) into Eq. (55) leads to

$$\xi^{n+1} = g \xi^n \quad (65)$$

where g denotes the growth factor, and from EQs. (65) and (55) we get

$$g^2 - 2g \left[1 - 2 \frac{c^2 \phi(\Delta t)^2}{\beta(\Delta x)^2} \sin^2 \left(\frac{k_x \beta(\Delta x)}{2} \right) - 2 \frac{c^2 \phi(\Delta t)^2}{\gamma(\Delta y)^2} \sin^2 \left(\frac{k_y \gamma(\Delta y)}{2} \right) \right] + 1 = 0 \quad (66)$$

We assume that the roots of equation are g_1 and g_2 . Now, we can prove that

$$\left| 1 - 2 \frac{c^2 \phi(\Delta t)^2}{\beta(\Delta x)^2} \sin^2 \left(\frac{k_x \beta(\Delta x)}{2} \right) - 2 \frac{c^2 \phi(\Delta t)^2}{\gamma(\Delta y)^2} \sin^2 \left(\frac{k_y \gamma(\Delta y)}{2} \right) \right| \leq 1 \quad (67)$$

$$0 \leq 2 \frac{c^2 \phi(\Delta t)^2}{\beta(\Delta x)^2} \sin^2 \left(\frac{k_x \beta(\Delta x)}{2} \right) + 2 \frac{c^2 \phi(\Delta t)^2}{\gamma(\Delta y)^2} \sin^2 \left(\frac{k_y \gamma(\Delta y)}{2} \right) \leq 2 \quad (68)$$

Hence,

$$\frac{c^2 \phi(\Delta t)^2}{\beta(\Delta x)^2} + \frac{c^2 \phi(\Delta t)^2}{\gamma(\Delta y)^2} \leq 1 \quad (69)$$

Then the equation (66) can be written as

$$g^2 - 2g \cos \zeta + 1 = 0$$

$$g = \cos \zeta \pm \sqrt{\cos^2 \zeta - 1} \quad (70)$$

Where

$$\cos \zeta = 1 - 2 \frac{c^2 \phi(\Delta t)^2}{\beta(\Delta x)^2} \sin^2 \left(\frac{k_x \beta(\Delta x)}{2} \right) - 2 \frac{c^2 \phi(\Delta t)^2}{\gamma(\Delta y)^2} \sin^2 \left(\frac{k_y \gamma(\Delta y)}{2} \right),$$

thus yields $|g_1| = |g_2| = 1$.

Therefore, the scheme is conditionally stable.

Local truncation error of finite difference scheme

Lemma (7): The local truncation error $\tau_{i,j}^n$ of the scheme (49) is $O((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2)$.

Proof. To estimate the truncation error of scheme (46), we perform a Taylor expansion. The local truncation error is given by:

$$\tau_{i,j}^n = \frac{1}{(\Delta t)^2} [u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}] - \frac{c^2}{(\Delta x)^2} [u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n] - \frac{c^2}{(\Delta y)^2} [u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n] \quad (71)$$

using Taylor's series expansion, we get

$$\frac{1}{(\Delta t)^2} (u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}) = \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n + \frac{(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n + \dots \quad (72)$$

$$\frac{1}{(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) = \left(\frac{\partial^2 u}{\partial x^2} \right)_{i,j}^n + \frac{(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n + \dots \quad (73)$$

$$\begin{aligned} \frac{1}{(\Delta y)^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) &= \left(\frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \\ \frac{(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n &+ \dots \end{aligned} \quad (74)$$

substitution from (72), (73) and (74) into (71)

$$\begin{aligned} \tau_{i,j}^n &= \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \\ \frac{(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n &- \frac{c^2 (\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \frac{c^2 (\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n \\ \dots & \end{aligned} \quad (75)$$

But is the solution of differential Eq. (2), so

$$\left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n = 0$$

Therefore, the foremost term in the local truncation error is given by

Then, the local truncation error

$$\begin{aligned} \frac{(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n &- \frac{c^2 (\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \\ \frac{c^2 (\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n &\dots \end{aligned}$$

$$\tau_{i,j}^n = O((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2) \quad (76)$$

Local truncation error of non-standard finite difference scheme

Lemma (8). The local truncation error $\tau_{i,j}^n$ of the scheme (55) is $O(\phi(\Delta t)^2 + \beta(\Delta x)^2 + \gamma(\Delta y)^2)$.

Proof. To estimate the truncation error of scheme (55), we perform out a Taylor expansion. The local truncation error is given by:

$$\begin{aligned} \tau_{i,j}^n &= \frac{1}{\phi(\Delta t)^2} [u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}] - \\ \frac{c^2}{\beta(\Delta x)^2} [u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n] &- \\ \frac{c^2}{\gamma(\Delta y)^2} [u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n] & \end{aligned} \quad (77)$$

using Taylor's series expansion, we get

$$\begin{aligned} \frac{1}{\phi(\Delta t)^2} (u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}) &= \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n + \\ \frac{\phi(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n &+ \dots \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{1}{\beta(\Delta x)^2} (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) &= \left(\frac{\partial^2 u}{\partial x^2} \right)_{i,j}^n + \\ \frac{\beta(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n &+ \dots \end{aligned} \quad (79)$$

$$\begin{aligned} \frac{1}{\gamma(\Delta y)^2} (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) &= \left(\frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \\ \frac{\gamma(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n &+ \dots \end{aligned} \quad (80)$$

substitution from (78), (79) and (80) into (77)

$$\begin{aligned} \tau_{i,j}^n &= \left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n + \\ \frac{\phi(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n &- \frac{c^2 \beta(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \\ \frac{c^2 \gamma(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n &\dots \end{aligned} \quad (81)$$

But is the solution of differential Eq. (2), so

$$\left(\frac{\partial^2 u}{\partial t^2} \right)_{i,j}^n - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n = 0$$

Therefore, the foremost term in the local truncation error is given by

$$\begin{aligned} \frac{\phi(\Delta t)^2}{12} \left(\frac{\partial^4 u}{\partial t^4} \right)_{i,j}^n &- \frac{c^2 \beta(\Delta x)^2}{12} \left(\frac{\partial^4 u}{\partial x^4} \right)_{i,j}^n - \\ \frac{c^2 \gamma(\Delta y)^2}{12} \left(\frac{\partial^4 u}{\partial y^4} \right)_{i,j}^n &\dots \end{aligned}$$

Then, the local truncation error is

$$\tau_{i,j}^n = O(\phi(\Delta t)^2 + \beta(\Delta x)^2 + \gamma(\Delta y)^2) \quad (82)$$

Numerical Results

This section presents the numerical results for the two-dimensional wave equation using Test Problem (2). Both the FDM and NSFDM were employed to

solve the partial differential equation, verifying the schemes' accuracy and computational efficiency. The absolute error, defined as the difference between the numerical and analytical solutions, was used as a measure of accuracy. The results are summarized and presented in Tables [3–6].

Test Problem (2):

The two-dimensional wave equation is expressed as

$$u_{tt}(x, y, t) = c^2[u_{xx}(x, y, t) + u_{yy}(x, y, t)]$$

$$\text{Where } x, y \in [0, 1]$$

The boundary conditions

$$u(0, y, t) = u(1, y, t) = 0, \quad \forall t > 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad \forall t > 0,$$

with initial conditions

$$u(x, y, 0) = f(x, y),$$

$$u_t(x, y, 0) = g(x, y),$$

The analytical solution is: (Daileda, 2014)

$$u(x, y, t) = \cos(\sqrt{2\pi}ct) \sin(\pi x) \sin(\pi y).$$

Let use steps sizes $h = 0.05$, $k = 0.0001$, $M = 500$, $N = 20$ and $c = 1$. Now, we use the FDM and the NSFDM to find numerical solutions for this example.

Table (3): Analytical and numerical results at $y=0.5$ using FDM

x_i	Analytical solution	FDM	Absolute error
0	0	0	0
0.05	0.15259	0.152606	0.0000155057
0.1	0.301424	0.301454	0.0000306297
0.15	0.442835	0.44288	0.0000449994
0.2	0.573342	0.5734	0.0000582611
0.25	0.689731	0.689801	0.0000700882
0.3	0.789137	0.789217	0.0000801896
0.35	0.869112	0.8692	0.0000883163
0.4	0.927686	0.927781	0.0000942685
0.45	0.963418	0.963516	0.0000978994
0.5	0.975427	0.975526	0.0000991197
0.55	0.963418	0.963516	0.0000978994
0.6	0.927686	0.927781	0.0000942685
0.65	0.869112	0.8692	0.0000883163
0.7	0.789137	0.789217	0.0000801896
0.75	0.689731	0.689801	0.0000700882
0.8	0.573342	0.5734	0.0000582611
0.85	0.442835	0.44288	0.0000449994
0.9	0.301424	0.301454	0.0000306297
0.95	0.15259	0.152606	0.0000155057
1	1.19455×10^{-16}	0	1.19455×10^{-16}

Table (3) gives the approximate solution for this example at $y = 0.5$ and its comparison to the analytical solution. Includes the absolute error. The maximum value of error for this example in the interval $[0, 1]$ is

$$l_{\infty} = 0.000099119$$

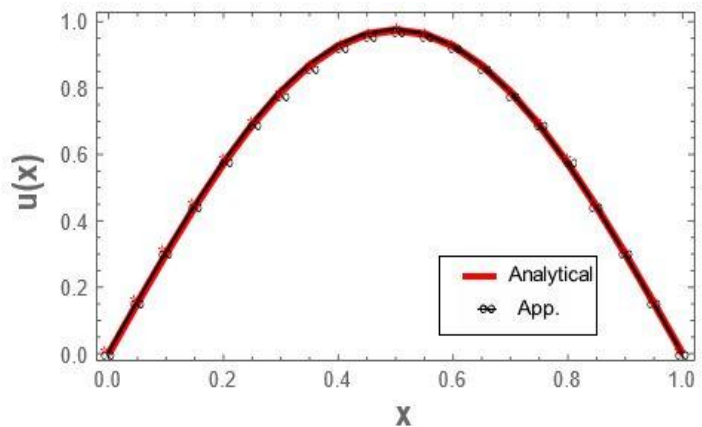


Fig. (5): The numerical outcomes and the analytical solutions by FDM at $y=0.5$

Table (4): Analytical and numerical results at $y=0.5$ using NSFDM

x_i	Analytical solution	NS-FDM	Absolute error
0	0	0	0
0.05	0.15259	0.152603	0.0000123268
0.1	0.301424	0.301448	0.0000243501
0.15	0.442835	0.44287	0.0000357739
0.2	0.573342	0.573388	0.0000463167
0.25	0.689731	0.689787	0.0000557191
0.3	0.789137	0.789201	0.0000637495
0.35	0.869112	0.869182	0.0000702101
0.4	0.927686	0.927761	0.000074942
0.45	0.963418	0.963496	0.0000778285
0.5	0.975427	0.975506	0.0000787987
0.55	0.963418	0.963496	0.0000778285
0.6	0.927686	0.927761	0.000074942
0.65	0.869112	0.869182	0.0000702101
0.7	0.789137	0.789201	0.0000637495
0.75	0.689731	0.689787	0.0000557191
0.8	0.573342	0.573388	0.0000463167
0.85	0.442835	0.44287	0.0000357739
0.9	0.301424	0.301448	0.0000243501
0.95	0.15259	0.152603	0.0000123268
1	1.19455×10^{-16}	0	1.19455×10^{-16}

Table (4) gives the approximate solution for this example at $y = 0.5$ and its comparison to the analytical solution. Includes the absolute error. The maximum value of error for this example in the interval $[0, 1]$ is

$$l_{\infty} = 0.0000787987$$

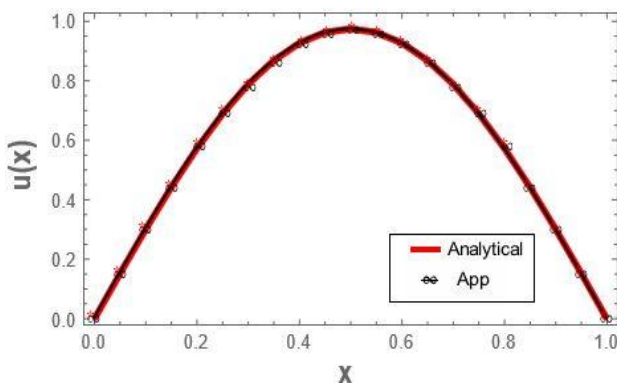


Fig. (6): The numerical outcomes and the analytical solutions by NSFDM at $y=0.5$

In Figures [5, 6], for $x \in [0,1]$ at $y=0.5$, the numerical results are represented by a black curve, while analytical solutions are shown as a red curve. We conclude that the analytical and numerical solutions obtained by NSFDM are very close, which gives more accuracy to the solution.

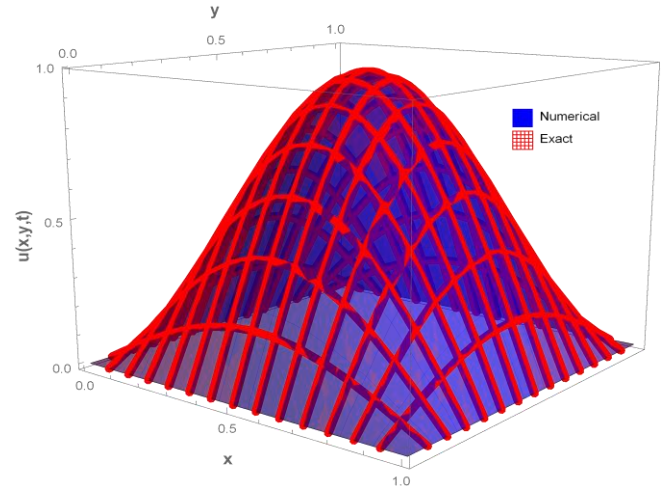


Fig. (7): The numerical outcomes and the analytical solutions by FDM

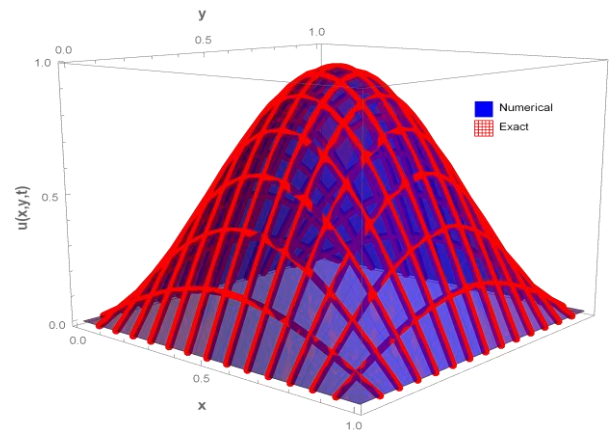


Fig. (8): The numerical outcomes and the analytical solutions by NSFDM

In Figures [7, 8], within the interval $[0,1]$, the numerical results are represented by a blue solid surface, while the analytical solutions are shown as a transparent surface with red mesh. We conclude that the analytical and numerical solutions obtained by NSFDM are very close, which gives more accuracy to the solution.

In case of $k=0.01$ and $h=0.05$.

Table (5): Analytical and numerical results at $k=0.01$ and $y=0.5$ using FDM

x_i	Analytical solution	FDM	Absolute error
0	0	0	0
0.05	0.15259	0.153362	0.000771567
0.1	0.301424	0.302948	0.00152414
0.15	0.442835	0.445074	0.00223918
0.2	0.573342	0.576241	0.00289908
0.25	0.689731	0.693219	0.0034876
0.3	0.789137	0.793127	0.00399024
0.35	0.869112	0.873507	0.00439463
0.4	0.927686	0.932377	0.00469081
0.45	0.963418	0.96829	0.00487148
0.5	0.975427	0.980359	0.00493221
0.55	0.963418	0.96829	0.00487148
0.6	0.927686	0.932377	0.00469081
0.65	0.869112	0.873507	0.00439463
0.7	0.789137	0.793127	0.00399024
0.75	0.689731	0.693219	0.0034876
0.8	0.573342	0.576241	0.00289908
0.85	0.442835	0.445074	0.00223918
0.9	0.301424	0.302948	0.00152414
0.95	0.15259	0.153362	0.000771567
1.0	1.19455×10^{-16}	0	1.19455×10^{-16}

Table (6): Analytical and numerical results at $k=0.01$ and $y=0.5$ using NSFDM

x_i	Analytical solution	NS-FDM	Absolute error
0	0	0	0
0.05	0.15259	0.15336	0.000769115
0.1	0.301424	0.302943	0.00151929
0.15	0.442835	0.445067	0.00223206
0.2	0.573342	0.576232	0.00288987
0.25	0.689731	0.693208	0.00347651
0.3	0.789137	0.793115	0.00397756
0.35	0.869112	0.873493	0.00438066
0.4	0.927686	0.932362	0.0046759
0.45	0.963418	0.968274	0.004856
0.5	0.975427	0.980344	0.00491653
0.55	0.963418	0.968274	0.004856
0.6	0.927686	0.932362	0.0046759
0.65	0.869112	0.873493	0.00438066
0.7	0.789137	0.793115	0.00397756
0.75	0.689731	0.693208	0.00347651
0.8	0.573342	0.576232	0.00288987
0.85	0.442835	0.445067	0.00223206
0.9	0.301424	0.302943	0.00151929
0.95	0.15259	0.15336	0.000769115
1.0	1.19455×10^{-16}	0	1.19455×10^{-16}

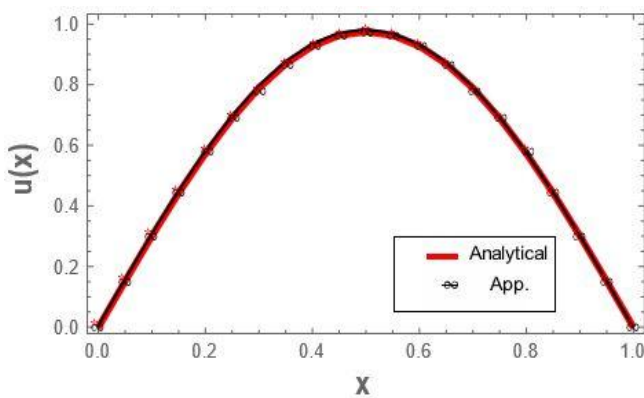


Fig. (9): The numerical outcomes and the analytical solutions at $k=0.01$ and $y=0.5$ by FDM

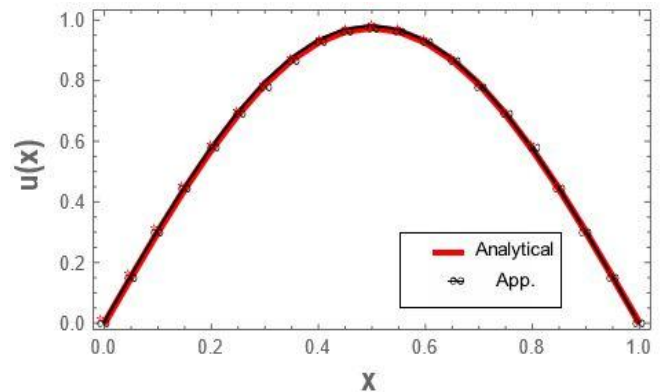


Fig. (10): The numerical outcomes and the analytical solutions at $k=0.01$ and $y=0.5$ by NSFDM

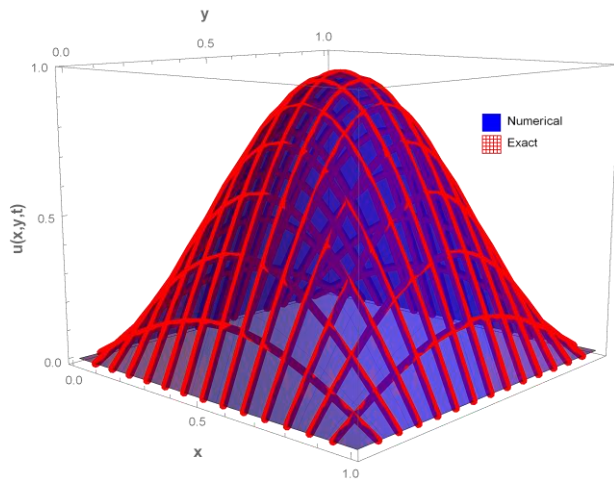


Fig. (11): The numerical outcomes and the analytical solutions at $k=0.01$ by FDM

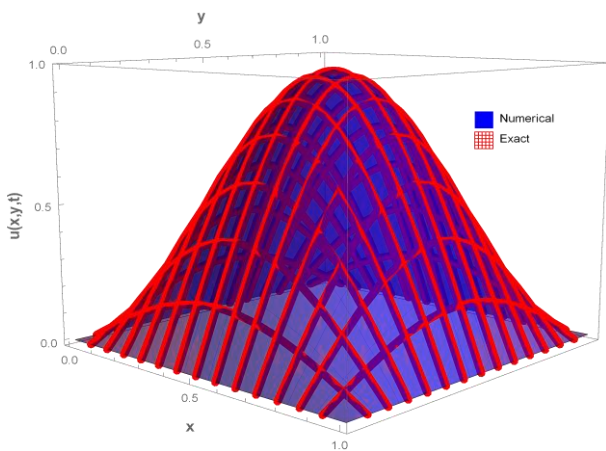


Fig. (12): The numerical outcomes and the analytical solutions at $k=0.01$ by NSFDM

Conclusion

In this paper, the exact and numerical solutions for Test Problems 1 and 2 are presented using the proposed methods. By comparing the finite difference method with the non-standard finite difference method, we conclude that the non-standard finite difference method achieves higher accuracy. The absolute error analysis demonstrates that our method provides a reliable approximation for solving the two-dimensional heat and wave equations using the finite difference approach. Additionally, the stability analysis of both methods, conducted using Von-Neumann theory, reveals that they are

conditionally stable when applied to the two-dimensional heat and wave equations.

Acknowledgements

The authors extend their gratitude to the reviewers for their helpful comments.

Conflict of interest

The authors declare that there are no conflicts of interest concerning this work.

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المحاكاة العددية للمعادلات التفاضلية الجزئية ثنائية الأبعاد

عبد المقصود عبد القادر سليمان^١، كمال رسلان محمد رسلان^٢، خالد كرم^٢، أحمد صبيح عواد^{١*}

^١قسم الرياضيات، كلية العلوم، جامعة العريش، مصر

^٢قسم الرياضيات، كلية العلوم، جامعة الأزهر، مصر

هذه الورقة مخصصة لتطوير مخططات جديدة لحل معادلة الحرارة ثنائية الأبعاد ومعادلة الموجة ثنائية الأبعاد بطريقة الفروق المحدودة وطريقة الفروق المحدودة غير القياسية. تمت مناقشة تحليل الاستقرار للمخططين وتوفير شروط استقرار مشروطة. تم حساب أخطاء الاقتران المحلية لكلا المخططين، وبالتالي تأكيد اتساقهما وتقارب المخططين. تم الحصول على نتائج عددية لحالتين اختبار مختلفتين لكل معادلة ومقارنتها بالحلول الدقيقة والنتائج العددية الأخرى التي تكشف عن الأخطاء المطلقة وتولد جداول التقارب. تُظهر النتائج الرسومية السلوك المكاني والزمني للحلول، مع التأكيد على فعالية المخططات المقدمة. عندما نقارن طريقة الفروق المحدودة بطريقة الفروق المحدودة غير القياسية، فإن تقنية طريقة الفروق المحدودة غير القياسية تتفوق على الأخيرة من حيث التخفيف من الآثار الرقمية. تقدم هذه الورقة منهجية شاملة لحل معادلات التفاضل الجزئي متعددة الأبعاد مع الحفاظ على الدقة النظرية (الاستقرار، وتحليل الأخطاء) والتحقق العملي (المعايير العددية)، مع تطبيقات في الفيزياء الحسابية والهندسة.