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Research Article **Mathematics**

Rough Continuous Functions on Information Systems

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ABSTRACT

The idea of rough continuous functions is essential to comprehending the dynamics and structure of data in the field of information systems theory. Within the information systems paradigm, this study examines the connection between rough set theory and continuity. We start by outlining the rough sets' theoretical underpinnings and how they are used in data analysis and display. Building upon this foundation, we introduce the notion of rough continuous functions, which extend traditional continuity concepts to accommodate the ambiguity intrinsic to rough set theory. Key contributions include a formal definition of rough continuous functions and an exploration of their properties within information systems. We investigate how these functions facilitate a more nuanced understanding of data behavior, particularly in scenarios where data exhibit varying degrees of granularity and uncertainty. Practical implications are discussed, highlighting the potential applications of rough continuous functions in data-driven decision-making and information system design. By connecting rough set theory with continuous functions, this paper not only advances theoretical understanding but also offers practical insights into managing and interpreting complex data structures within information systems.

Introduction

Rough set theory [brief. (RST)], introduced by **(Pawlak, 1982)**, is a mathematical approach to deal with uncertainty and vagueness in data analysis **(Kandil et al., 2013; Mashhour, 1970; Salama and Abd El-Monsef, 2011b; Lashin et al., 2005; Salama and Abd El-Monsef, 2011a; Hai Yu and Zhan, 2014; Abu-Donia and Salama, 2012; Slowinski, 2000; Wybraniec Skardowska, 1989; Yao, 1998)**. Unlike classical set theory, which requires precise information, (RST) provides a framework for working with imprecise or incomplete information. It is particularly useful for classifying objects in situations where data may be inconsistent or incomplete. One of the earliest approaches to data analysis that was not statistical, its methodology focuses on the classification and examination of vague, unclear, or insufficient knowledge and data **(Pawlak, 1982)**. The approximation of a set's upper and lower boundaries, which is the formal classification of knowledge about the topic of interest, is the basic idea behind (RST) **(Kandil et al., 2016)**.

The key concept of rough sets revolves around the approximation of sets. A rough set is defined by two approximations: the lower approximation set [brief. (LAS)], which contains all objects that surely belong to the set, and the upper approximation set [brief. (UAS)], which includes all objects that possibly belong to the set. The difference between these approximations reflects the uncertainty or "roughness" of the data. Throughout time, the theory of Rough Set has demonstrated itself to be a helpful tool for addressing a variety of problems, such as expressing ambiguous or uncertain knowledge, knowledge analysis, evaluating the consistency and the reliability of information based on its availability and date patterns, recognizing and evaluating date dependency, and reasoning based on ambiguous and reduced data **(Salama, 2010; 2020b; 2011a; 2014a)**.

Rough set applications are employed far more widely now than they were formerly, mostly in the fields of process control, evaluation of database characteristics, and medical **(Salama, 2008b; 2014b; 2016; 2020a)**. This paper covers (RST), information systems [brief. (IS)], continuous functions on (IS), and topological functions. It can be defined as a function that maintains certain attributes or connections between data elements throughout system transformations or states.

In this paper, we aim to explore rough continuity in greater depth and propose novel ideas regarding rough functions within information systems. We begin by discussing the fundamental concepts in **Section (1)**. **Section (2)** introduces the (IS) and the information table [brief. (IT)]. **Section (3)** Clarifying the Connection to Information Systems. **Section (4)** details the methods used in our study, followed by a presentation and explanation of rough continuous functions on (IS) in **Section (5)**. The paper concludes with the conclusions drawn in **Section (6)** and suggestions for future work in **Section (7)**. An Ethics Statement is provided separately, and References are included at the end of the manuscript.

Fundamental concepts

For many years, (RST) has been evolving, and an increasing number of scholars are becoming interested in its methods. It is a formal theory that emerged from basic studies of (IS) logical characteristics. In relational databases, (RST) has been utilized for database mining and knowledge discovery for a considerable time. This part discusses the concepts of (RST), which are in some ways like the concepts of other theories that deal with ambiguous and unclear data. Two of the most popular and traditional techniques for addressing and modeling uncertainties now include the Dempster-Shafer theory of uncertainty and Fuzzy set theory **(Pawlak et al., 1995)**. The subsequent discussion highlights the central concepts of (RST):

Sets

A group of entities having Shared characteristics are a fundamental element in mathematical concepts. Mathematical entities like relations, functions, and integers can all be considered as sets. Nevertheless, the idea regarding conventional sets in mathematics is paradoxical because a set is seen as a "categorization" In which no elements are included, and it is defined as an empty set **(Stoll, 1979)**. A pertinence connection is a relationship that exists between an element and a set. A set's constituent pieces are referred to as its elements. The cardinality of a set is a measure of its elemental count. Examples of sets that handle ambiguous and uncertain data are provided below:

a. Fuzzy set

In the last half of the **1960s**, mathematician **Loft Zadeh** created it with the intention of handling the imprecise and approximate mathematical concept for computer programming and storage. For Zadeh to arrive at the mathematical foundation for fuzzy set theory [brief. (FST)], he had to return to the classical theory of sets, where each set is described by a function. To identify values as members of the Universal Set $\boldsymbol{U}$ that fall inside the real number $[0,1]$ interval **(Abu-Donia and Salama, 2010; Qin and Zheng, 2005; Zhao Tsang, 2008; Li et al., 2008, Gong et al., 2008)**, the characteristic function can be extended to the fuzzy category.

The fuzzy set A and the function referred to as the Function of pertinence characterize the fuzzy function as, the set $\mu_A: U \to [0,1]$ is defined in such a way that, $\mu_A(a) =$ the degree to which $a \in A$ **(Zadeh, 1965)**, $\mu_A(a) = 1$, meaning a is fully in the set A . $\mu_A(a) = 0$, meaning a is not in the set A. $0 < \mu_A(a) < 1$, this means a partially belongs to the set A, with the value indicating the degree of membership.

b. Rough set

A method that mathematician **(Pawlak, 1982)** first proposed at the start of the **1980s**; it is a technique used in mathematics for managing imprecise and nebulous data. Like The boundary area of a set conveys uncertainty and imprecision in (FST) and (RST). rather than by a set's incomplete participation, as in (FST) does. The notion of a rough set is broadly defined by approximations known as topological procedures, both interior and closure.

Observation

Studying fuzzy, rough, and classical sets and contrasting their interpretations is fascinating **(Kandil et al., 2011; Salama, 2008a; Liu and Sai, 2009; Abu-Donia, 2008)**. The definition of a classical set is axiomatic or intuitive, and it is a primitive idea. Fuzzy sets are characterized by the fuzzy membership function, which employs complex mathematical structures, integers, and functions. Since topological processes known as approximations define the rough set, this definition also necessitates a deep understanding of mathematics.

Information system (IS) and information table (IT)

An (IS) or data table can be thought of as a grid where the rows represent objects, and the columns represent their attributes. **(Salama and El Barbary, 2017; Abu-Donia et al., 2007; Kryszkiewicz, 1998; Srinivasan et al., 2001; El Barbary et al., 2017)**. It is applied to the data format that Rough Set will adopt, in which a set number of characteristics are assigned to each object **(Lin, 1997)**. These items are detailed based on the layout of the data table, with columns representing attributes and rows corresponding to the objects being analyzed **(Wu et al., 2004)**. The following is an example of an information table:

Table (1): Demonstration of an (IT)

Patients	**Attributes**			
	Vomiting	**Temperature**	**Headache**	**Viral illness**
P1	Yes	High	No	Yes
P2	No	High	Yes	Yes
P3	Yes	Very High	Yes	Yes
P4	Yes	Normal	No	No
P5	No	High	Yes	No
P6	Yes	Very High	No	Yes

Relation of indiscernibility

An essential concept in (RST) is the indiscernibility relation [brief. (IR)], which is defined as a relationship between two or more objects in which every value is the same with respect to a subset of the qualities under consideration. An equivalency connection known as the "indiscernibility relation" regards all identical objects in a collection as elementary **(Pawlak, 1998)**. It is evident from Table (1), which is discussed in Section (4), This set comprises characteristics closely connected to the patients' symptoms, including vomiting, temperature, and headache. Upon dissecting Table (1), it becomes evident that the set pertaining to {patient 2, patient 3, patient 5} is indistinguishable with respect to the property of headache. When it comes to the vomiting attribute, the set pertaining to {patient 1, patient 3, patient 4} is undetectable. Patient 2 and

Patient 5 are similar in terms of vomiting, temperature, and headache; however, Patient 2 has a viral disease whereas Patient 5 does not. Consequently, the patients' set comprising patient 2 and patient 5 have unresolved symptoms.

Approximations

(RST) begins with (IR), which is based on data about objects of interest. This relation represents the concept that, based on the available information, it is not possible to differentiate between certain objects due to a lack of knowledge. Another key idea in (RST) is approximations, which are related to the interpretation of topological operations that approximate something **(Wu et al., 2004)**. In the topology generated by the (IR), Upper and lower approximations of a set are analogous to closure and interior operations. The different types of approximations in (RST) are described and clarified below.

a. Lower Approximation Set (LAS) $\underline{A}$

(LAS) defines the domain objects that can be confidently determined to be part of the subset of interest. For a set N in relation to R, this approximation includes all objects that are unmistakably categorized as belonging to N based on R. This is denoted as $\underline{A}$.

b. Upper Approximation Set (UAS) $\overline{A}$

(UAS) refers to the objects that might be included in the subset of interest. For a set M in relation to R, the (UAS) consists of all objects that could potentially be classified as part of M according to R. This set is denoted as $\overline{A}$.

c. Boundary Region (BN)

The Boundary Region describes the objects of a set X concerning R that cannot be definitively classified as either X or its complement X^c with respect to R. If the Boundary Region is empty ϕ, the set is termed "Crisp", meaning it is precisely defined in relation to R. Conversely, if the (BN) is not empty $X \neq \phi$, the set X is considered "Rough". Mathematically, the (BN) is represented as $BR = \overline{A} - \underline{A}$. Given a set $X \subseteq U$, an equivalence relation A, and a knowledge base $K = (U, A)$.

Two subsets can be correlated:

1. A-lower: $\underline{A} = \cup\{Y \in U/A : Y \subseteq X\}$
2. A -upper: $\overline{A} = \cup\{Y \in U/A : Y \cap X \neq \emptyset\}$.

In a comparable fashion, $POS(A), BN(A)$ and $NEG(A)$ are defined as follows **(Pawlak, 1991)**.

3. $POS(A) = \underline{A} \Rightarrow$ definitely member of X.
4. $NEG(A) = U - \overline{A} \Rightarrow$ certainly non-member of X.
5. $BR(A) = \overline{A} - \underline{A} \Rightarrow$ potentially member of X.

Figure (1) illustrates these regions visually **(Lambert-Torres et al., 1999)**.

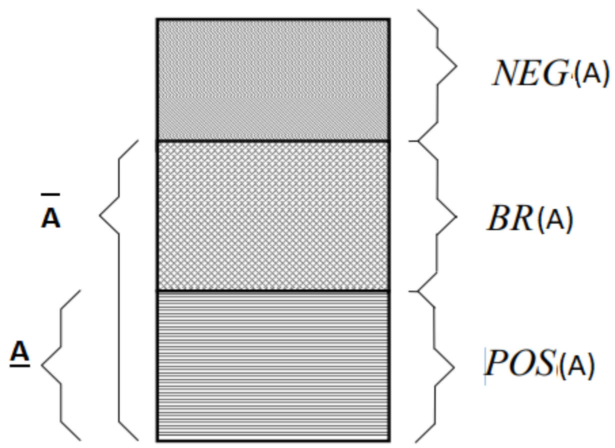

Fig. (1): Explanation of A-approximation sets and A-regions

d. Quality Approximation

It is calculated numerically using its own components, particularly those of lower and upper approximations. The coefficient used to assess quality is denoted by $\alpha B(Y)$, where Y refers to a set of objects or records related to B. The quality of approximation is assessed using two coefficients described below. One of these coefficients is the Imprecision Coefficient, $\alpha A(Y)$, which measures the approximation quality of X and is represented by:

$$\alpha A(Y) = |\underline{A}(Y)|/|\overline{A}(Y)| \ldots\ldots\ldots (1)$$

Where $|\underline{A}(Y)|$ and $|\underline{A}(Y)|$ it represents the cardinality of approximation lower and upper, and the approximation are set $\neq \phi$. Therefore, $0 \leq \alpha A \leq 1$, If $\alpha A(Y) = 1$, then Y is a well-defined set with respect to the attributes A, meaning that Y is a crisp set. If $\alpha A(Y) < 1$, then Y is a rough set with respect to the attributes A. Applying this to Table (1), $\alpha A(Y) = 3/5$ for patients who might have a viral illness. The other coefficient is the Quality Coefficient of the upper and lower approximations. The Quality Coefficient of the upper approximation, $\alpha A(\overline{A}(Y))$ represents the proportion of all elements classified as belonging to Y and is denoted by:

$$\alpha A\left(\overline{A}(Y)\right) = \frac{|\overline{A}(Y)|}{|A|} \ldots\ldots\ldots (2)$$

In Table (1), $\alpha A\left(\overline{A}(Y)\right) = 5/6$ for patients who might have a viral illness. The Quality Coefficient of the lower approximation, $\alpha A(\underline{A}(Y))$ represents the percentage of all elements that are likely classified as belonging to y and is denoted by:

$$\alpha A\left(\underline{A}(Y)\right) = \frac{|\underline{A}(Y)|}{|A|} \ldots\ldots\ldots (3)$$

In Table (1), $\alpha A(\underline{A}(Y)) = 3/6 = 1/2$, for patients with a viral illness.

Note: In the Quality Coefficients for the upper and lower approximations presented in point 2 of this section, $|A|$ denotes the cardinality of a given set of objects **(Herawan, 2015)**.

Decision tables and decision algorithms

In a decision table [brief. (DT)], attributes fall into two categories: condition attributes and decision attributes. In Table (1), as outlined in Section (3), attributes like vomiting, temperature, and headache are classified as condition attributes, while viral illness is categorized as a decision attribute. Each row in the decision table outlines the actions or decisions to be made when the conditions specified by the condition attributes are fulfilled. For example, in Table (1), the conditions (vomiting: yes), (headache: no), and (temperature: high) result in the decision (viral illness: yes). In Table (1), the condition attributes of headache, vomiting, and temperature have the same values, indicating that patients 2 and 5 present the same symptoms. Nevertheless, the decision attribute values differ among these patients. These regulations are said to as contradictory, non-determinant, or inconsistent. These guidelines are referred to as determinate, consistent, non-conflicting, or just guidelines. A factor of consistency, represented by $\gamma(C, D)$, where C is the condition and D is the decision, is the quantity of consistency rules that are present in the decision table. The decision table is inconsistent if $\gamma(C, D) \neq 1$; otherwise, it is consistent if $\gamma(C, D) = 1$. Considering that Table (1) has $\gamma(C, D) = 4/6$, meaning that, Within the set of six rules for Table (1), there are two inconsistent rules (covering patients 2, 5, and 6) and four consistent rules (covering patients 1, 3, 4, and 6). Decision rules are commonly expressed in the form of "if... then..." statements. One guideline is provided for the implied viral disease in order to proceed:

If
Vomiting = yes
and
Temperature = high
and
Headache = no
Then
Viral Illness = yes

If a fever is high, vomiting is present, and headaches are absent, then viral illness is present.

Decision algorithms are sets of decision rules associated with (DT). These algorithms encompass all the rules outlined in the corresponding (DT). It is important to distinguish between decision tables and decision algorithms. A decision algorithm, which consists of logical expressions, differs from a (DT), which is a collection of implications **(Pawlak, 1991)**. For additional details on contemporary rough set methodologies and applications, see **(Pomykala, 1987; El Barbary and Salama, 2019; Qin et al., 2008; Al-shami, 2017; Al-shami and Noiri, 2019; Zhu, 2007; Zhu, 2009; Yao, 1998a; Yao, 1998b; Bonikowski, 1998)**.

Clarifying the Connection to Information Systems

Rough set theory is a fundamental tool in data analysis and uncertainty management within information systems. This theory relies on upper and lower approximations to define imprecise or ambiguous sets, making it particularly valuable in fields such as data analysis, decision-making, and intelligent system design. This study aims to explore the continuity of rough functions within information systems, drawing connections between continuity in topological spaces and the approximation processes used in data analysis. The research introduces a mathematical definition of rough continuous functions, examines their properties, and demonstrates their application in handling uncertain and evolving data across various domains.

Methods

Study Aim: The aim of this study is to explore and analyze the properties of rough continuous functions within the framework of information systems. We focus on theoretical aspects and mathematical proofs to understand how rough continuity interacts with various structures in information systems.

Design and Setting: This research is purely theoretical and is conducted through mathematical analysis and logical reasoning. The study is based on established theories and models in the field of information systems, with an emphasis on theoretical exploration rather than empirical data.

Procedures: We investigated rough continuous functions by applying mathematical proofs and theoretical constructions. The study involves examining definitions, properties, and examples of rough continuous functions as they pertain to information systems. We utilized logical reasoning and theoretical frameworks from existing literature to analyze these functions.

Statistical Analysis: Since this is a theoretical study, statistical analysis is not applicable. The findings are derived through mathematical proofs and theoretical exploration.

Rough continuous functions on information systems

Throughout this section, consider $\mathcal{S}_1 = (\mathcal{U}_1, \mathcal{A}_1, \mathcal{V}_1)$ and $\mathcal{S}_2 = (\mathcal{U}_2, \mathcal{A}_2, \mathcal{V}_2)$ are two information systems that have been topologized. This paper aims to define the concept of IS-rough continuity and to explore its various characteristics.

Definition (1): The map $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is continuous if and only if it preserves the structure of the information in a certain manner.

This definition can be grounded in the set of attributes and how values are distributed among objects in the two systems. In this context, the continuous function is viewed as preserving the relationship between objects and their attributes across the systems.

Requirements:

a. Attribute Preservation:

There should be a relationship between the attributes in the two systems. If $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$ are corresponding attributes, their values under the influence of function f must be compatible.

b. Value Preservation:

If there is a relationship between attribute values in the two systems, the values should transform consistently through the function f. In other words, if $v(a, x)$ represents the value of attribute a for object x in $\mathcal{S}_1$, and $v'(b, f(x))$ represents the value of attribute b for object $f(x)$ in $\mathcal{S}_2$, then they should satisfy:

$$v(a, x) = v'(b, f(x))$$

or there should be an appropriate transformation function.

c. Open and Closed Sets:

If a topological structure exists within the systems, the function f must preserve the characteristics of open and closed sets. Specifically, if a set is open in the first system, then its preimage under f should be open in the second system.

Definition (2): The function $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is *rough continuous* is considered rough continuous if the preimage of every open set in $\mathcal{U}_2$ is an open set in $\mathcal{U}_1$ i.e., $f^{-1}(A_2) \subseteq \mathcal{U}_1$ is open for every attribute A_2 in $\mathcal{U}_2$ be open.

Definition (3): Let $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a rough continuous function. The lower approximation set in $\mathcal{S}_1$ is mapped to the lower approximation set in $\mathcal{S}_2$, **i.e.** $f(\underline{A}) = \underline{f(A)}$.

Definition (4): Let $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a *rough continuous function*. The upper approximation set in $\mathcal{S}_1$ is mapped to the upper approximation set in $\mathcal{S}_2$, **i.e.** $f(\overline{A}) = \overline{f(A)}$.

Example (1): Let S_1 and S_2 are two systems, such that S_1 represent a Company Data and S_2 refer Employee Data and assume we have a function f that maps the attributes of the companies to the attributes of the employees as follow:

Step (1): Constructing Indiscernibility Tables

Information System S_1 (Industry type and Company size):

Entities	Industry type	Company size
C_1	Technology	Large
C_2	Healthcare	Medium
C_3	Finance	Small
C_4	Technology	Medium
C_5	Healthcare	Large

Information System $\mathcal{S}_2$ (Salary and Experience):

Entities	Salary	Experience
J_1	High	Senior
J_2	Medium	Mid- level
J_3	Low	Beginner
J_4	Medium	Senior
J_5	High	Mid- level

Step (2): Function $f: \mathcal{S}_1 \to \mathcal{S}_2$ Calculating Lower and Upper Approximations

The function f maps each company to a typical job position within that company:

Company (in $\mathcal{S}_1$)	*Mapped Job position* (in $\mathcal{S}_2$)
C_1	J_1
C_2	J_2
C_3	J_3
C_4	J_4
C_5	J_5

Step (3): Rough Continuity analysis Calculating Lower and Upper Approximations

Lower approximation Example:

Subset $A_2 \subseteq U_2$: Consider $A_2 = \{J_1, J_5\}$, representing high salary position.

The lower bound approximation $\underline{A_2}$:

The job position set that is unequivocally included in A_2 based on the attributes. For simplicity, assume $\underline{A_2} = \{J_1, J_5\}$ (i.e., the exact match).

Inverse Image $f^{-1}(A_2)$ in $\mathcal{S}_1$: Companies that map to A_2.

$f^{-1}(A_2) = \{C_1, C_5\}$ (Large Technology and Large Healthcare companies)

The Lower Approximation $\underline{f^{-1}(A_2)}$ in $\mathcal{S}_1$: The set of companies that belong to $f^{-1}(A_2)$, i.e., $\underline{f^{-1}(A_2)} = \{C_1, C_5\}$ (Exact match).

Check Rough Continuity:

$$f^{-1}\left(\underline{A_2}\right) = \{C_1, C_5\} = \underline{f^{-1}(A_2)}$$

The function f preserves the lower approximation, showing rough continuity.

Upper approximation Example:

Subset $A_2 \subseteq U_2$: Consider $A_2 = \{J_2, J_4\}$, representing Medium-salary position.

The Upper bound approximation $\overline{A_2}$: The job positions set that could potentially be part of A_2 . Assume $\overline{A_2} = \{J_2, J_4\}$.

Inverse Image $f^{-1}(A_2)$ in $\mathcal{S}_1$: Companies that map to A_2.

$f^{-1}(A_2) = \{C_2, C_4\}$

(Medium Healthcare and Medium Technology companies)

The Upper Approximation $\overline{f^{-1}(A_2)}$ in $\mathcal{S}_1$: The set of companies that possibly belong to $f^{-1}(A_2)$, i.e., $\overline{f^{-1}(A_2)} = \{C_2, C_4\}$ (Exact match).

Check Rough Continuity:

$$f^{-1}\left(\overline{A_2}\right) = \{C_2, C_4\} = \overline{f^{-1}(A_2)}$$

The function f preserves the upper approximation, confirming rough continuity.

Theorem (1): Let $f: \mathcal{S}_1 \to \mathcal{S}_2$ be a function from a topologized Information system $\mathcal{S}_1 = (\mathcal{U}_1, \mathcal{V}_1, \mathcal{D}_1)$ to a topologized information system $\mathcal{S}_2 = (\mathcal{U}_2, V_2, D_2)$. Therefore, the following statements are interchangeable:

(1) f is considered rough continuous.

(2) The preimage of every lower approximation set in $\mathcal{S}_2$ be lower approximation set in $\mathcal{S}_1$.

(3) The preimage of every upper approximation set in $\mathcal{S}_2$ be upper approximation set in $\mathcal{S}_1$

Proof (1): Implies (2) and (3) Assume that f is rough continuous. By the definition of rough continuity, f preserves the rough structure between the systems $\mathcal{S}_1$ and $\mathcal{S}_2$. Specifically, this means that for any set $N \subseteq \mathcal{S}_2$, $f^{-1}(\underline{N}) = \underline{f^{-1}(N)}$ and $f^{-1}(\overline{N}) = \overline{f^{-1}(N)}$. Since $\underline{f^{-1}(N)}$ be a lower approximation is the set in $\mathcal{S}_1$, it follows that the preimage image of a lower approximation set in $\mathcal{S}_2$ is a lower approximation set in $\mathcal{S}_1$. Similarly, since $\overline{f^{-1}(N)}$ is an upper approximation set in $\mathcal{S}_1$, the preimage image of an upper approximation set in $\mathcal{S}_2$ is an upper approximation set in $\mathcal{S}_1$. This shows that $f^{-1}(\underline{N})$ is a lower approximation set in $\mathcal{S}_1$ and $f^{-1}(\overline{N})$ is an upper approximation set in $\mathcal{S}_1$. Thus, (2) and (3) hold.

(2) and (3) implies (1) Assume that the preimage of every lower approximation set in $\mathcal{S}_2$ is a lower approximation set in $\mathcal{S}_1$, and the preimage of every upper approximation set in $\mathcal{S}_2$ is an upper approximation set in $\mathcal{S}_1$. We need to show that f is rough continuous, meaning:

$f^{-1}(\underline{M}) = \underline{f^{-1}(M)}$ and $f^{-1}(\overline{M}) = \overline{f^{-1}(M)}$. Let $M \subseteq \mathcal{S}_2$. Since $f^{-1}(\underline{M})$ is a lower approximation set in $\mathcal{S}_1$, and $\underline{f^{-1}(M)}$ is the greatest lower approximation set in $\mathcal{S}_1$ contained in $f^{-1}(M)$, we have

$\underline{f^{-1}(M)} \subseteq f^{-1}(\underline{M})$. However, by the assumption that $f^{-1}(M)$ is a lower approximation set in $\mathcal{S}_1$ and the definition of $\underline{f^{-1}(M)}$, we also have: $f^{-1}(\underline{M}) \subseteq \underline{f^{-1}(M)}$. Hence, $\underline{f^{-1}(M)} = f^{-1}(\underline{M})$. A similar argument applies to upper approximation sets, showing that: $f^{-1}(\overline{X}) = \overline{f^{-1}(X)}$.

This proves that f is rough continuous.

(2) Implies (3) Assume that the preimage of every lower approximation set in $\mathcal{S}_2$ is a lower approximation set in $\mathcal{S}_1$. We want to prove that the preimage of every upper approximation set in $\mathcal{S}_2$ is an upper approximation set in $\mathcal{S}_1$. Given

$B \subseteq \mathcal{S}_2$, we know that $\overline{B} = \left(\underline{(B^c)}\right)^c$. Take the preimage under f of both sides $f^{-1}(\overline{B}) = f^{-1}\left(\left(\underline{(B^c)}\right)^c\right)$. We know that the preimage of the complement of a set is the complement of the inverse image of the set $f^{-1}\left(\left(\underline{(B^c)}\right)^c\right) = \left(f^{-1}\underline{(B^c)}\right)^c$. Thus, $f^{-1}(\overline{B}) = \left(f^{-1}\underline{(B^c)}\right)^c$. By assumption (2), $f^{-1}\underline{(B^c)}$ is a lower approximation set in $\mathcal{S}_1$. Therefore, $f^{-1}(\overline{B}) = \underline{\left(f^{-1}(B^c)\right)^c}$. But the complement of a lower approximation set is, by definition, an upper approximation set. Therefore, $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$. This shows that the preimage of an upper approximation set in $\mathcal{S}_2$ is an upper approximation set in $\mathcal{S}_1$, which is exactly statement (3).

(3) Implies (2), Assume that the preimage of each upper approximation set in $\mathcal{S}_2$ is an upper approximation set in $\mathcal{S}_1$. We need to prove that the preimage of each lower approximation set in $\mathcal{S}_2$ is a lower approximation set in $\mathcal{S}_1$. Given $V \subseteq \mathcal{S}_2$, we know that $\underline{V} = \left(\overline{V^c}\right)^c$. Take the preimage under f of both sides $f^{-1}(\underline{V}) = f^{-1}\left(\left(\overline{V^c}\right)^c\right)$. Using the property that the inverse image of the complement of a set is the complement of the preimage of the set $f^{-1}\left(\left(\overline{V^c}\right)^c\right) = \left(f^{-1}\overline{(V^c)}\right)^c$. Thus, $f^{-1}(\underline{V}) = \left(f^{-1}\overline{(V^c)}\right)^c$. By assumption (3), $f^{-1}\overline{(V^c)}$ is an upper approximation set in $\mathcal{S}_1$. Therefore, $f^{-1}(\underline{V}) = \left(\overline{f^{-1}(V^c)}\right)^c$. But the complement of an upper approximation set is, by definition, a lower approximation set. Therefore, $f^{-1}(\underline{V}) = \underline{f^{-1}(V)}$. This shows that the preimage of a lower approximation set in $\mathcal{S}_2$ is a lower approximation set in $\mathcal{S}_1$, which is exactly statement (2) and completing the equivalence proof.■

Theorem (2): Let $\mathcal{S}_1 = (\mathcal{U}_1, V_1, D_1)$ and $\mathcal{S}_2 = (\mathcal{U}_2, V_2, D_2)$ be two information systems. If $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is rough continuous, then the preimage of each exact set in $\mathcal{S}_2$ is exact set in $\mathcal{S}_1$.

Proof. Assume $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is rough continuous. Let $N \subseteq \mathcal{S}_2$ be an exact set in $\mathcal{S}_2$. By the definition of exactness, this means $\underline{N} = \overline{N}$. We need to show that the inverse image $f^{-1}(N) \subseteq \mathcal{S}_1$ is exact in $\mathcal{S}_1$, meaning $\underline{f^{-1}(N)} = \overline{f^{-1}(N)}$. Since f is rough continuous, we have the following two equalities:
$f^{-1}(\underline{N}) = \underline{f^{-1}(N)}$

$f^{-1}(\overline{N}) = \overline{f^{-1}(N)}$

Because N is exact in $\mathcal{S}_2$, we know that $\underline{N} = \overline{N}$. Substituting this into the rough continuity conditions, we get $f^{-1}(\underline{N}) = f^{-1}(\overline{N})$. But from the rough continuity conditions, we also have $\underline{f^{-1}(N)} = \overline{f^{-1}(N)}$.

Since $\underline{f^{-1}(N)} = \overline{f^{-1}(N)}$, the set $f^{-1}(N)$ is exact in $\mathcal{S}_1$. ■

Theorem (3): Let $\mathcal{S}_1 = (\mathcal{U}_1, V_1, D_1)$, $\mathcal{S}_2 = (\mathcal{U}_2, V_2, D_2)$ and $\mathcal{S}_3 = (\mathcal{U}_3, V_3, D_3)$ be three information systems. If $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ and $g: \mathcal{S}_2 \longrightarrow \mathcal{S}_3$ are two rough continuous functions, then $g \circ f: \mathcal{S}_1 \longrightarrow \mathcal{S}_3$ is rough continuous.

Proof. Let M be a lower approximation set in $\mathcal{S}_3$. Since g is rough continuous function, then we have $g^{-1}(M)$ is a lower approximation set in $\mathcal{S}_2$. Since f is rough continuous function, then $f^{-1}(g^{-1}(M)) = (f^{-1} \circ g^{-1})(M) = (g \circ f)^{-1}(M)$

be a lower approximation set in $\mathcal{S}_1$. Therefore $g \circ f$ is rough continuous. ■

Remark (1): Let $\mathcal{S}_1 = (\mathcal{U}_1, V_1, D_1)$ and $\mathcal{S}_2 = (\mathcal{U}_2, V_2, D_2)$ be two information systems and let $f: \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a function. Then

(1) If A_1 is a lower approximation set in $\mathcal{S}_1$, every subset of A_1 is a lower approximation set in $\mathcal{S}_1$, then f is rough continuous.

(2) If $\mathcal{S}_2$ and ϕ are the only lower approximation sets in $\mathcal{S}_2$, then f is rough continuous.

Proof:

(1) Given that each subset of A_1 is a lower approximation set in $\mathcal{S}_1$, it follows that for any set $V \subseteq \mathcal{S}_2$, the inverse image $f^{-1}(V) \subseteq \mathcal{S}_1$ must also be a lower approximation set in $\mathcal{S}_1$. Specifically, this means $f^{-1}(V) = \underline{f^{-1}(V)}$. Now, consider the lower approximation set $\underline{V}$ in $\mathcal{S}_2$. The rough continuity of f requires that $f^{-1}(\underline{V}) = \underline{f^{-1}(V)}$.

Since $f^{-1}(V) = \underline{f^{-1}(V)}$ for any set $V \subseteq \mathcal{S}_2$, it follows that $f^{-1}(\underline{V}) = f^{-1}(V)$. Thus, $f^{-1}(\underline{V})$ is indeed equal to $\underline{f^{-1}(V)}$, satisfying the condition for rough continuity. The assumption that every subset of A_1 is a lower approximation set in $\mathcal{S}_1$ ensures that f is rough continuous.

(2) Assume that the only lower approximation sets in $\mathcal{S}_2$ are $\mathcal{S}_2$ (i.e., the entire space U_2) and ϕ. For any set $B \subseteq U_2$, $\underline{X}$ must be either U_2 or ϕ. If $\underline{B} = U_2$, then $B = U_2$ (since the lower approximation includes all elements). If $\underline{B} = \phi$, then B must be a subset of some indiscernible or empty set. Consider the inverse image under f. If $\underline{B} = U_2$, then $f^{-1}(\underline{B}) = f^{-1}(U_2) = U_1$, and since U_1 is the whole space in $\mathcal{S}_1$, $\underline{f^{-1}(X)} = U_1$. If $\underline{B} = \phi$, then $f^{-1}(\underline{B}) = f^{-1}(\phi) = \phi$, and $\underline{f^{-1}(B)} = \phi$. In both cases, the condition for rough continuity holds $f^{-1}(\underline{B}) = \underline{f^{-1}(B)}$. Since $f^{-1}(\underline{B}) = \underline{f^{-1}(B)}$ is true for any $B \subseteq U_2$, the function f is rough continuous.■

Theorem (4): Let $\mathcal{S}_1 = (\mathcal{U}_1, V_1, D_1)$ and $\mathcal{S}_2 = (\mathcal{U}_2, V_2, D_2)$ be two information systems. If $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is a constant function, then f is rough continuous**.**

Proof. Let $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a constant function such that $f(v) = b \in \mathcal{S}_2$ for every element v in $\mathcal{S}_1$. Let N be a lower approximation set in $\mathcal{S}_2$. If $b \notin N$, then $f^{-1}(N) = \phi$ which is a lower approximation set in $\mathcal{S}_1$. But if $b \in N$, then $f^{-1}(b) = \mathcal{S}_1$ which is also a lower approximation set in $\mathcal{S}_1$. Therefore f is rough continuous. ■

Remark (2): Let A and B be two lower approximation sets in an information system $\mathcal{S}_1 = (\mathcal{U}_1, V_1, D_1)$. Then

(1) $A \cap B$ is a lower approximation set.

(2) $A \cap B$ is a lower approximation set.

Proof:

(1) Let A and B be two lower approximation sets in $\mathcal{S}_1$. Then $\underline{A} = A$ and $\underline{B} = B$. we have $\underline{A \cap B} = \underline{A} \cap \underline{B}$. Then $\underline{A \cap B} = A \cap B$. Hence $A \cap B$ is a lower approximation set in $\mathcal{S}_1$.

(2) Let A and B be two a lower approximation sets in $\mathcal{S}_1$, then A and B are open sets. Then $A \cup B$ is open set in $\mathcal{S}_1$, that is $A \cup B = (A \cup B)^{\circ} = \underline{A \cup B}$. Hence $A \cup B$ is a lower approximation set in $\mathcal{S}_1$. ■

Conclusion

This study introduces and investigates the concept of rough continuous functions within the framework of information systems, allowing for a more flexible and precise model for analyzing uncertain data. The findings show that these functions preserve the core structure of rough set theory, providing a powerful tool for data analysis in various fields such as decision-making, pattern recognition, and machine learning.

Future work

While this paper lays the groundwork for understanding rough continuous functions in information systems, several avenues for future research remain open:

a. **Expanding Applications:** Exploring the use of rough continuous functions in areas such as medical diagnosis, financial forecasting, and intelligent systems support.
b. **Algorithm Development:** Designing efficient algorithms for computing rough continuous functions in large-scale information systems, optimizing computational complexity.
c. **Integration with Other Theories:** Combining rough continuous functions with theories such as fuzzy logic and artificial neural networks to develop more robust models for managing ambiguous data.
d. **Empirical Validation:** Testing the application of rough continuous functions on real-world datasets to evaluate their performance compared to traditional methods.

Through these advancements, rough continuous functions can play a crucial role in data analysis and decision-making within advanced information systems.

Ethics Statement

This research is purely theoretical and does not involve human participants, animal subjects, or sensitive data that requires ethical review. Therefore, an ethics statement is not applicable.

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"الدوال التقريبية المستمرة في نظم المعلومات"

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فكرة الدوال الخشنة المستمرة ضرورية لفهم ديناميكيات وهيكل البيانات في مجال نظرية نظم المعلومات. تبحث هذه الدراسة داخل نموذج نظم المعلومات عن العلاقة بين نظرية المجموعات الخشنة والاستمرارية. نبدأ بتوضيح الأسس النظرية للمجموعات الخشنة واستخداماتها في تحليل البيانات وتمثيلها. بناءً على هذه الأسس، نقدم مفهوم الدوال الخشنة المستمرة التي توسع مفاهيم الاستمرارية التقليدية لتتكيف مع الغموض المتأصل في نظرية المجموعات الخشنة. تشمل المساهمات الرئيسية تعريفًا رياضيًا للدوال الخشنة المستمرة واستكشاف خصائصها داخل نظم المعلومات. نتحقق من كيفية تسهيل هذه الدوال فهمًا أكثر دقة لسلوك البيانات، خاصة في السيناريوهات التي تظهر فيها البيانات درجات متفاوتة من الحبيبية وعدم اليقين. تُناقش التطبيقات العملية، مع إبراز الاستخدامات المحتملة للدوال الخشنة المستمرة في صنع القرار القائم على البيانات وتصميم نظم المعلومات. من خلال الربط بين نظرية المجموعات الخشنة والدوال المستمرة، لا يقدم هذا البحث تقدمًا في الفهم النظري فحسب، بل يقدم أيضًا رؤى عملية لإدارة وتفسير هياكل البيانات المعقدة داخل نظم المعلومات.