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MATHEMATICS

Generalized Coordinated Search Problem for a Randomly Located Target

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ABSTRACT

In this paper, we aim to obtain the investigation of the coordinated search problem, in which the velocities of each searcher is a random variable. Our main goal is to find a randomly located target on the real line; this target may be a hole in an oil pipeline or a cut in a power cable below the sea surface. Currently, there are two searchers on the line, and instead of starting at the origin, they search from any point on the line. The first searcher searches the right part, and the second searcher searches the left part. We provide the most efficient coordinated search method, and we calculate the expected value of the time to detect the target. In addition, the conditions under which this expected value can be minimized were derived. The most important aspect of this method is that we have possession of the most effective method for search optimization.

Introduction

Searching for a missing target is considered a momentous issue at all times, and significant applications of the probability theory, such as searching for the explosive devices, the volcanoes, the fire beams, the exploration submarine, the remains of sunk ships, and the treasures in the depths of the seas and the oceans, etc. The missing target may be stationary or moving and has symmetric or asymmetric distribution.

The main purpose is to find this missing target in the shortest amount of time (with the lowest possible cost). In the case of a randomly located target on the real line, which has a known probability distribution, the searcher will search for the target at a certain speed according to the nature of the search environment. The target will be found if the searcher reaches it. This problem has been extensively studied in many previous works (**Mohamed and Abou Gabal, 2000; El-Rayes et al. 2003; Mohamed, 2005; Stone et al. 2016; Afifi and El-Bagoury, 2021; Afifi and Kacem, 2024**).

One of the most well-known methods of searching for a lost target is the coordinated search method. It represents one of the top proposed look methods, with the search range partitioned into specific cells or areas to search. It depends on two searchers, S_1 and S_2 starting from the origin and looking for the lost target: S_1 and S_2, searching for the target on the right and left parts, respectively. They return to the starting point after searching for the missing target.

This method was previously investigated by Reyniers (1995) when the target had a known bounded symmetric distribution and was hidden on an interval. For cases when the missing target has uniform, triangular, or truncated exponential distributions, Reyniers found a generic optimality condition and applied it to the problem. Furthermore, this method was studied for symmetric continuous distributing targets on the line by (**Reyniers, 1996**). When the target is continuously and decreasing, Reyniers provided the necessary and sufficient conditions for the presence of an optimal search method.

Teamah et al. (2007) have recently examined this problem on the line when the target has an asymmetric distribution, they have an optimal search method to locate the targets that are asymmetrically distributed; based on their findings, they also show that some of the previous studies were special cases from their results. Interesting methods of search that provide the lowest anticipated value of the costs of finding the missing target have also been studied. Several authors, **Mohamed et al. 2009; Mohamed et al. 2012; El-Hadidy and Abou-Gabal, 2015; El-Hadidy and El-Bagoury, 2017; El-Hadidy and Elshenawy, 2023; Alamri and El-Hadidy, 2024** have focused on the problem of a randomly located target in the plane where the target has symmetric and asymmetric distributions, and the searchers have less information than they have available. The authors of these works aimed to minimize the anticipated time required to identify the target.

Elbery, 2007 additionally discussed the problem of coordinated search for a randomly hiding target on one of two lines. In the other situation of the coordinated search problem, two searchers search for a missing target on the line at random locations. At certain points in the line, they start their search together. The distribution of the missing target is symmetric around the point where the searcher's movement starts, which is the intersection of two lines. Four searchers start their coordinated search for the missing target at the intersection. For this case, they developed the search method and computed the anticipated value of the searcher-target first meeting time. This case was discussed by **Teamah et al. (2018)**.

Another case combines the linear search problem with the coordinated search problem. In this case, two searchers are considered to begin searching for the target together from the point where the two lines intersect ($\boldsymbol{a_o} = \boldsymbol{b_o} = \mathbf{0}$). By searching at the areas on the first and second lines, respectively, each searcher aims to find the missing target on his line. This case was discussed by **Abou Gabal, 2018**.

In the previous articles mentioned, the searcher's path was deterministic, but **Al-freedi and El-Hadidy, (2019)** presented the minimum expected value of the detecting time for the oil pipeline hole under sea surface. He uses the linear coordinated search method with two searchers starting their search from the origin. In this paper, we will generalize the previous search method. The generalization here means that searchers start their search at any arbitrary point. We aim to determine the optimal search method that reduces the expected value of the detection time.

Problem Formulation

We describe this problem as follows

A. The Searching Framework

Searching space: The real line $\mathbf{L}$ (oil pipeline or power cable under sea surface).

The target: The target is randomly located in the real line $\mathbf{L}$ with a known probability distribution.

The search: The searching process is done by two searchers, $\mathbf{S_1}$ and $\mathbf{S_2}$ who start their search from any point on $\mathbf{L}$. The right and left sides of the starting point on the line will be searched by $\mathbf{S_1}$ and $\mathbf{S_2}$, respectively.

B. The Searching Plan

The searchers will start the searching process together at the same moment from starting point $a_0 = b_0$, and then there exist two cases: the first when $a_0 > 0$, $\mathbf{S_1}$ will cover the right side, while $\mathbf{S_2}$ will cover the left side.

Both of them have the ability to communicate (using modern means of communication) with a ship located at the starting point to facilitate the process of communication. They don't return to the starting point as in the previous works.

The method of searching will be as follows:

1) $\mathbf{S_1}$ starts from the point $\mathbf{a_0}$ and goes to the right part of $\mathbf{a_0}$ as far as $c_1 = a_1 - a_0$ where $0 < c_1 \le k_1^2$, then sends a report to the ship whether the target is founded or not, he waits for the reply from the ship to complete the searching process or not.
2) If the response was to not complete the searching, then the target has been found by the searcher $\mathbf{S_2}$. Otherwise, go to step III.
3) In case the target is not found then, $\mathbf{S_1}$ continues in the same direction as far as $\mathbf{c_2} = \mathbf{a_2} - \mathbf{a_1}$, then $\mathbf{S_1}$ sends the report to the ship. These steps are repeated until the target is found (see Figs. 1, 2).
4) The searcher $\mathbf{S_2}$ will follow the same search method, but on the left side of the starting point.

The probability density function

$$h(V) = |V - V_0|\delta(V^2 - V_0), \quad -\infty \le V \le \infty, \tag{1}$$

is the probability density function of the random velocity of any searcher, where v_0 is the initial velocity, and δ is the Dirac Delta function see **Teamah et al. (2012).**

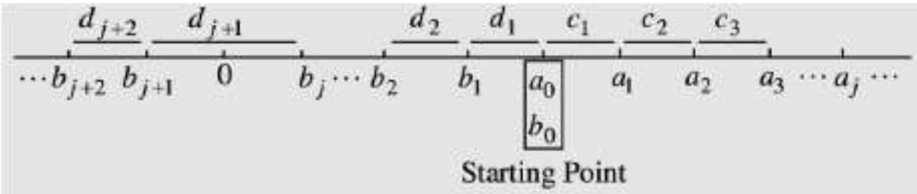

Fig. (1): The starting point in the right part $a_0 = b_0 > 0$.

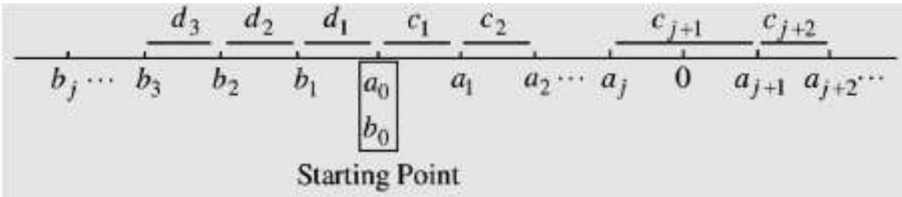

Fig. (2): The starting point in the left part $a_0 = b_0 < 0$.

We consider the PDF of the random distance ci, di is given by:

$$g_1(c_i) = \frac{1}{k_i\sqrt{c_i}} - \frac{1}{k_i^2} \quad , \quad 0 < c_i \le k_i^2. \tag{2}$$

And,

$$g_2(d_i) = \frac{1}{\tilde{k_i}\sqrt{d_i}} - \frac{1}{\tilde{k_i^2}} \quad , \quad 0 < d_i \le k_i^2 \tag{3}$$

Since we will consider that in the optimal case, the maximum distance traveled by $\mathbf{S_1}$ in the right part before the first contact will be k_i (in the left part will be $\tilde{k}_i$). And the distances traveled by the searchers are random because the search process takes place under the sea surface. And those distances are $0 < c_i \le k_i^2$ in the right part, $c_i = a_i - a_{i-1}$ (and $0 < d_i \le \widetilde{k_i^2}$ in the left part where $d_i = b_i - b_{i-1}$).

We assume that the position of the target is represented by the random variable X which has a certain asymmetric distribution on the line, leads to the distances covered by each searcher are different depending on the possibility of the target being on the line.

Let the random variable $Z \ge 0$ has a distribution with expected value $\mathbf{E(Z)}$ then we can assume that $(k_i < \tilde{k_i})$ or $(k_i > \tilde{k_i})$, then we can consider $\tilde{k_i} = k_i + E(Z)$.

Therefore, the equation (3) become

$$g_2(d_i) = \frac{1}{(k_i + E(Z))\sqrt{d_i}} - \frac{1}{(k_i + E(Z))^2},$$

$$0 < d_i \le (k_i + E(Z))^2. \tag{4}$$

To detect the target, the searcher $\mathbf{S_1}$ follows search path c which defined by a sequence $\{c_i; i \ge 1\}$. Also, the searcher $\mathbf{S_2}$ follows path d which defined by a sequence $\{d_i; i \ge 1\}$. Then the search plan be defined by $\phi = (c, d) \in \Phi$ (the set of all search plans).

Define

$$\alpha = inf\{x; F(x) > 0\}, \text{and } \beta = sup\{x; F(x) < 1\}, \tag{5}$$

where $\mathbf{F(x)}$ is the distribution function of the target position, α and β is the minimum and maximum value of b_i and c_i respectively. Let γ be the probability measure resulting from target position where $\gamma(x,y) = F(y) - F(x)$. Also let the target detecting time be $\tau(\varphi)$.

Theorem (1): The expected value of the time of finding the target is given by the following equation:

$$E(\tau(\phi)) = \sum_{i=0}^{\infty}\left[\frac{k_{i+1}^4 - 4a_0^{3/2}k_{i+1} + 3a_0^2}{6k_{i+1}^2}\right][\gamma(\alpha, b_i)]$$

$$+\sum_{i=0}^{j-1}\left[\frac{(k_{i+1}+E(Z))^4 - 4b_0^{3/2}(k_{i+1}+E(Z)) + 3b_0^2}{6(k_{i+1}+E(Z))^2}\right][\gamma(a_i, \beta)]$$

$$-\left[\sum_{i=j}^{\infty}\left(\frac{(k_{i+1}+E(Z))^4 - 4b_0^{3/2}(k_{i+1}+E(Z)) + 3b_0^2}{6(k_{i+1}+E(Z))^2}\right)(\gamma(a_i, \beta))\right].$$

Proof. Since, the searchers S_1 and S_2 shift on the pipe with random distances (as shown by the two relationships (2) and (4)) depending on the probability of the target being on the line. Thus, the expected value of the distance in the right and is the left respectively,

$$E(c_i) = \int_{a_0}^{k_i^2} c_i\left(\frac{1}{k_i\sqrt{c_i}} - \frac{1}{k_i^2}\right)d\,c_i = \frac{k_i^2}{6} - \frac{2a_0^{3/2}}{3k_i} + \frac{a_0^2}{2k_i^2},$$

$$E(d_i) = \frac{(k_i+E(Z))^2}{6} - \frac{2b_0^{3/2}}{3(k_i+E(Z))} + \frac{b_0^2}{2(k_i+E(Z))^2},$$

let, the expected value of the velocity of the searchers is given by $E(V) = \pm 1$, based on the above equations, the expected value of the time of finding the target in the right part is:

$$\tau_1 = \frac{E(c_i)}{+1} = \frac{k_i^4 - 4a_0^{\frac{3}{2}}k_i + 3a_0^2}{6k_i^2},$$

and in the left part is:

$$\tau = \begin{cases} \dfrac{E(d_i)}{+1} = \dfrac{(k_i+E(z))^4 - 4b_0^{\frac{3}{2}}(k_i+E(Z)) + 3b_0^2}{6(k_i+E(Z))^2}, & 0 \le b_j < b_0 \\ \dfrac{E(d_i)}{-1} = -\dfrac{(k_i+E(z))^4 - 4b_0^{\frac{3}{2}}(k_i+E(Z)) + 3b_0^2}{6(k_i+E(Z))^2}, & b_{j+1} < 0. \end{cases}$$

If the target lies in $]\,a_0, a_1]$, then

$$\tau_2 = \left[\frac{(k_1+E(Z))^4 - 4b_0^{3/2}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2}\right].$$

And if the target lies in $]\,a_{j-1}, a_j]$, then

$$\tau_2 = \left[\frac{(k_1+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2} + \frac{(k_2+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_2+E(Z)) + 3b_0^2}{6(k_2+E(Z))^2} + \cdots + \frac{(k_j+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_j+E(Z)) + 3b_0^2}{6(k_j+E(Z))^2}\right].$$

Thus, if the target lies in $]\,a_j, a_{j+1}]$, then

$$\tau_2 = \left[\frac{(k_1+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2} + \cdots + \frac{(k_j+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_j+E(Z)) + 3b_0^2}{6(k_j+E(Z))^2} - \left(\frac{(k_{j+1}+E(Z))^4 - 4b_0^{\frac{3}{2}}(k_{j+1}+E(Z)) + 3b_0^2}{6(k_{j+1}+E(Z))^2}\right)\right].$$

Similarly, if the target lies in $]\,b_{j+1}, b_j]$ then,

$$\tau_1 = \left[\frac{k_1^4 - 4a_0^{\frac{3}{2}}k_1 + 3a_0^2}{6k_1^2} + \frac{k_2^4 - 4a_0^{\frac{3}{2}}k_2 + 3a_0^2}{6k_2^2} + \cdots + \frac{k_{j+1}^4 - 4a_0^{\frac{3}{2}}k_{j+1} + 3a_0^2}{6k_{j+1}^2}\right]$$

and so on.

And because the target is distributed to the line, we get the expected value of finding the target with the following steps:

$$E(\tau(\phi)) = \left[\frac{(k_1+E(Z))^4 - 4b_0^{3/2}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2}\right][\gamma(a_0,a_1)]$$
$$+\left[\frac{(k_1+E(Z))^4 - 4b_0^{3/2}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2}\right.$$

$$\left.+\frac{(k_2+E(Z))^4 - 4b_0^{3/2}(k_2+E(Z)) + 3b_0^2}{6(k_2+E(Z))^2}\right][\gamma(a_1,a_2)]$$
$$+\left[\frac{(k_1+E(Z))^4 - 4b_0^{3/2}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2}\right.$$

$$+\frac{(k_2+E(Z))^4 - 4b_0^{3/2}(k_2+E(Z)) + 3b_0^2}{6(k_2+E(Z))^2}$$
$$\left.+\cdots+\frac{(k_j+E(Z))^4 - 4b_0^{3/2}(k_j+E(Z)) + 3b_0^2}{6(k_j+E(Z))^2}\right][\gamma(a_{j-1},a_j)]$$

$$+\left[\frac{(k_1+E(Z))^4 - 4b_0^{3/2}(k_1+E(Z)) + 3b_0^2}{6(k_1+E(Z))^2}+\cdots\right.$$
$$+\frac{(k_j+E(Z))^4 - 4b_0^{3/2}(k_j+E(Z)) + 3b_0^2}{6(k_j+E(Z))^2}$$

$$\left.-\left(\frac{(k_{j+1}+E(Z))^4 - 4b_0^{3/2}(k_{j+1}+E(Z)) + 3b_0^2}{6(k_{j+1}+E(Z))^2}\right)\right][\gamma(a_j,a_{j+1}]+\cdots$$
$$+\frac{k_1^4 - 4a_0^{3/2}k_1 + 3a_0^2}{6k_1^2}[\gamma(b_1,b_0)]$$

$$+\left[\frac{k_1^4 - 4a_0^{3/2}k_1 + 3a_0^2}{6k_1^2}+\frac{k_2^4 - 4a_0^{3/2} + 3a_0^2}{6k_2^2}\right][\gamma(b_2,b_1)]+\cdots$$
$$+\left[\frac{k_1^4 - 4a_0^{3/2}k_1 + 3a_0^2}{6k_1^2}\right.$$

$$\left.+\frac{k_2^4 - 4a_0^{3/2} + 3a_0^2}{6k_2^2}+\cdots+\frac{k_{j+1}^4 - 4a_0^{3/2}k_{j+1} + 3a_0^2}{6k_{j+1}^2}\right][\gamma(b_{j+1},b_j)]$$

$$= \frac{k_1^4 - 4a_0^{3/2}k_1 + 3a_0^2}{6k_1^2}\left[\gamma(b_1,b_0) + \gamma(b_2,b_1) + \cdots + \gamma(b_j,b_{j-1}) + \gamma(b_{j+1},b_j) + \ldots\right]$$

$$+ \frac{k_2^4 - 4a_0^{3/2}k_2 + 3a_0^2}{6k_2^2}\left[\gamma(b_2,b_1) + \cdots + \gamma(b_j,b_{j-1}) + \cdots + \gamma(b_{j+1},b_j) + \cdots\right] + \cdots$$

$$+ \frac{k_{j+1}^4 - 4a_0^{3/2}k_{j+1} + 3a_0^2}{6k_{j+1}^2}\left[\gamma(b_{j+1},b_j) + \cdots\right] + \cdots$$

$$+ \frac{(k_1 + E(Z))^4 - 4b_0^{3/2}(k_1 + E(Z)) + 3b_0^2}{6(k_1 + E(Z))^2}\left[\gamma(a_0,a_1) + \gamma(a_1,a_2) + \cdots + \gamma(a_{j-1},a_j) + \gamma(a_j,a_{j+1}) + \cdots\right]$$

$$+ \frac{(k_2 + E(Z))^4 - 4b_0^{3/2}(k_2 + E(Z)) + 3b_0^2}{6(k_2 + E(Z))^2}\left[\gamma(a_1,a_2) + \cdots + \gamma(a_{j-1},a_j) + \gamma(a_j,a_{j+1}) + \cdots\right] + \cdots$$

$$+ \frac{(k_j + E(Z))^4 - 4b_0^{3/2}(k_j + E(Z)) + 3b_0^2}{6(k_j + E(Z))^2}\left[\gamma(a_{j-1},a_j) + \cdots\right] - \left(\frac{(k_{j+1} + E(Z))^4 - 4b_0^{3/2}(k_{j+1} + E(Z)) + 3b_0^2}{6(k_{j+1} + E(Z))^2}\right)\left[\gamma(a_j,a_{j+1}) + \cdots\right] - \cdots$$

$$= \frac{k_1^4 - 4a_0^{3/2}k_1 + 3a_0^2}{6k_1^2}\left[\gamma(\alpha,b_0)\right] + \frac{k_2^4 - 4a_0^{3/2}k_2 + 3a_0^2}{6k_2^2}\left[\gamma(\alpha,b_1)\right] +$$

$$+ \frac{k_{j+1}^4 - 4a_0^{3/2}k_{j+1} + 3a_0^2}{6k_{j+1}^2}\left[\gamma(\alpha,b_j)\right] + \cdots + \frac{(k_1 + E(Z))^4 - 4b_0^{3/2}(k_1 + E(Z)) + 3b_0^2}{6(k_1 + E(Z))^2}\left[\gamma(a_0,\beta)\right]$$

$$+ \frac{(k_2 + E(Z))^4 - 4b_0^{3/2}(k_2 + E(Z)) + 3b_0^2}{6(k_2 + E(Z))^2}\left[\gamma(a_1,\beta)\right] + \cdots - \left(\frac{(k_{j+1} + E(Z))^4 - 4b_0^{3/2}(k_{j+1} + E(Z)) + 3b_0^2}{6(k_{j+1} + E(Z))^2}\right)\left[\gamma(a_j,\beta)\right] - \cdots$$

$$=\sum_{i=0}^{\infty}\left[\frac{k_{i+1}^4-4a_0^{3/2}k_{i+1}+3a_0^2}{6k_{i+1}^2}\right][\gamma(\alpha,b_i)]$$

$$+\sum_{i=0}^{j-1}\left[\frac{\left(k_{i+1}+E(Z)\right)^4-4b_0^{3/2}\left(k_{i+1}+E(Z)\right)+3b_0^2}{6\left(k_{i+1}+E(Z)\right)^2}\right][\gamma(a_i,\beta)]$$

$$-\left[\sum_{i=j}^{\infty}\left(\frac{(k_{i+1}+E(Z))^4-4b_0^{3/2}(k_{i+1}+E(Z))+3b_0^2}{6(k_{i+1}+E(Z))^2}\right)[\gamma(a_i,\beta)]\right] \quad (6)$$

Since, $0<a_i-a_{i-1}<k_i^2$, then if there exist $\varepsilon\geq 0$we get $k_i^2=(a_i-a_{i-1}+\varepsilon)^2$.

Hence, (6) take the form:

$$E\left(\tau(\phi)\right)=\sum_{i=0}^{\infty}\left[\frac{(a_{i+1}-a_i+\varepsilon)^4-4a_0^{3/2}(a_{i+1}-a_i+\varepsilon)+3a_0^2}{6(a_{i+1}-a_i+\varepsilon)^2}\right][\gamma(\alpha,b_i)]$$

$$+\sum_{i=0}^{j-1}\left[\frac{\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)^4-4b_0^{3/2}\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)+3b_0^2}{6\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)^2}\right][\gamma(a_i,\beta)]$$

$$-\left[\sum_{i=j}^{\infty}\left[\frac{\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)^4-4b_0^{3/2}\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)+3b_0^2}{6\left(a_{i+1}-a_i+\varepsilon+E(Z)\right)^2}\right][\gamma(a_i,\beta)]\right] \quad (7)$$

Also, if there exist $\xi\geq 0$ we get $\widetilde{k_i^2}=(b_i-b_{i-1}+\xi)^2$ .Compensate for $k_i=\widetilde{k_i}-E(Z)=b_i-b_{i-1}+\xi-E(Z)$ in (6) we get:

$$E\left(\tau(\phi)\right)=\sum_{i=0}^{\infty}\left[\frac{\left(b_{i+1}-b_i+\xi+E(Z)\right)^4-4a_0^{3/2}\left(b_{i+1}-b_i+\xi-E(Z)\right)+3a_0^2}{6\left(b_{i+1}-b_i+\xi-E(Z)\right)^2}\right][\gamma(\alpha,b_i)]$$

$$+\sum_{i=0}^{j-1}\left[\frac{(b_{i+1}-b_i+\xi)^4-4b_0^{3/2}(b_{i+1}-b_i+\xi)+3b_0^2}{6(b_{i+1}-b_i+\xi)^2}\right][\gamma(a_i,\beta)]$$

$$-\sum_{i=j}^{\infty}\left[\frac{(b_{i+1}-b_i+\xi)^4-4b_0^{3/2}(b_{i+1}-b_i+\xi)+3b_0^2}{6(b_{i+1}-b_i+\xi)^2}\right][\gamma(a_i,\beta)] \quad (8)$$

Existence of optimal search plan depends on finding the necessary and sufficient conditions. But because the target has bounded a symmetric distribution, we don't need to find these conditions. Directly, we can find the optimal values for the points ai and bi that give us the optimal path to find the target by using the equations (7) and (8).

Optimal Search Path for Bounded Asymmetric Target Distribution

Our aim in this section is to minimize $E(\tau(\varphi))$. This will happen by finding the optimal path values of $\{c_i, i \geq 1\}$ and $\{d_i, i \geq 1\}$ that will give us the optimal path for the given target distribution function from class Q. If this path is found, this will be the optimal search path.

We conclude that, if c^* and d^* are optimal values of c and d, respectively, and if Q^0 is a subclass of Q, then the optimal search path will be in Q^0. It clear that the search path depends on the distribution of the target $F(x)$ and the values of c and d, and they are unknown factors. And because the values of c and d depend on the values of $\{a_i, i \geq 0\}$ and $\{b_i, i \geq 0\}$, so we want to find the optimal values $\{a_i^*, i \geq 0\}$ and $\{b_i^*, i \geq 0\}$.

We assume that the target distribution $F(x)$ is known and regular (i.e., $F(x)$ is continuous with positive $f(x)$ and $E(x) < \infty$).

To get the optimal values, $\{a_i^*, i \geq 0\}$, you must solve this non-linear programming problem (NLP(1)):

NLP(1) :

$$\min_{a_j}\left[\frac{\left(a_j-a_{j-1}+\varepsilon\right)^4-4a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon\right)+3a_0^2}{6\left(a_j-a_{j-1}+\varepsilon\right)^2}\right]\left[\gamma\left(a,b_{j-1}\right)\right]$$

$$+\left[\frac{\left(a_{j+1}-a_j+\varepsilon\right)^4-4a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon\right)+3a_0^2}{6\left(a_{j+1}-a_j+\varepsilon\right)^2}\right]\left[\gamma\left(a,b_j\right)\right]$$

$$+\left[\frac{\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^4-4a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)+3a_0^2}{6\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^2}\right]\left[\gamma\left(a_{j-1},\beta\right)\right]$$

$$-\left[\frac{\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^4-4a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)+3a_0^2}{6\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^2}\right]\left[\gamma\left(a_j,\beta\right)\right]$$

subject to:

$$\frac{a_j-a_{j-1}+\varepsilon}{a_j-a_{j-1}}\geq 1,\quad \frac{a_{j+1}-a_j+\varepsilon}{a_{j+1}-a_j}\geq 1,\quad a_j-a_{j-1}>0,$$

$$a_{j+1}-a_j>0,\qquad \frac{a_j-a_{j-1}+\varepsilon+E(z)}{b_j-b_{j-1}}\geq 1,$$

$$\frac{a_{j+1}-a_j+\varepsilon+E(z)}{b_{j+1}-b_j}\geq 1,\quad b_j-b_{j-1}>0,\quad b_{j+1}-b_j>0,$$

$$\varepsilon\geq 0,\qquad z\geq 0.$$

Definition (1): (Optimal Solution) If there exist $a \in R$ such that $f(a^*) \leq f(a)$ for all $a \in R$, then $a^* \in R$ be the optimal solution of the NLP (1).

NLP (1) take the form:

NLP(2) :

$$\min_{a_j}\left[\frac{\left(a_j-a_{j-1}+\varepsilon\right)^4-4a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon\right)+3a_0^2}{6\left(a_j-a_{j-1}+\varepsilon\right)^2}\right]\left[\gamma\left(a,b_{j-1}\right)\right]$$

$$+\left[\frac{\left(a_{j+1}-a_j+\varepsilon\right)^4-4a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon\right)+3a_0^2}{6\left(a_{j+1}-a_j+\varepsilon\right)^2}\right]\left[\gamma\left(a,b_j\right)\right]$$

$$+\left[\frac{\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^4-4a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)+3a_0^2}{6\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^2}\right]\left[\gamma\left(a_{j-1},\beta\right)\right]$$

$$-\left[\frac{\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^4-4a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)+3a_0^2}{6\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^2}\right]\left[\gamma\left(a_j,\beta\right)\right]$$

$$+\left[\frac{1}{6}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^2-\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-1}+\frac{1}{2}a_0^2\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-2}\right]f\left(a_j\right)]$$

subject to:

$$1-\frac{a_j-a_{j-1}+\varepsilon}{a_j-a_{j-1}}\le 0,\quad 1-\frac{a_{j+1}-a_j+\varepsilon}{a_{j+1}-a_j}\le 0,$$

$$a_{j-1}-a_j<0,\qquad a_j-a_{j+1}<0,$$

$$1-\frac{a_j-a_{j-1}+\varepsilon+E(z)}{b_j-b_{j-1}}\le 0,$$

$$1-\frac{a_{j+1}-a_j+\varepsilon+E(z)}{b_{j+1}-b_j}\le 0,$$

$$b_{j-1}-b_j<0,\quad b_j-b_{j+1}<0,$$

$$-\varepsilon\le 0,\qquad -z\le 0.$$

By using Kuhn-Tucker condition, we find

$$\left[\begin{matrix}\frac{1}{3}\left(a_j-a_{j-1}+\varepsilon\right)+\frac{2}{3}a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon\right)^{-2}\\ -a_0^2\left(a_j-a_{j-1}+\varepsilon\right)^{-3}\end{matrix}\right]\left[\gamma\left(\alpha,b_{j-1}\right)\right]$$

$$+\left[\frac{-1}{3}\left(a_{j+1}-a_j+\varepsilon\right)-\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon\right)^{-2}+a_0^2\left(a_{j+1}-a_j+\varepsilon\right)^{-3}\right]\left[\gamma\left(\alpha,b_j\right)\right]$$

$$+\left[\frac{1}{3}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)+\frac{2}{3}a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^{-2}-a_0^2\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^{-3}\right]\left[\gamma\left(a_{j-1},\beta\right)\right]$$

$$-\left[\left[\frac{-1}{3}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)-\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-2}+a_0^2\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-3}\right]\left[\gamma\left(a_j,\beta\right)\right]\right.$$

$$+u_1\left(\frac{\epsilon}{\left(a_j-a_{j-1}\right)^2}\right)+u_2\left(\frac{-\epsilon}{\left(a_{j+1}-a_j\right)}\right)+u_3\left(\frac{-1}{\left(b_j-b_{j-1}\right)}\right)+u_4\left(\frac{1}{\left(b_{j+1}-b_j\right)}\right)+u_5(-1)+u_6(1)=0, \tag{9}$$

$$u_1\left(1-\frac{a_j-a_{j-1}+\varepsilon}{a_j-a_{j-1}}\right)=0, \tag{10}$$

$$u_2\left(1-\frac{a_{j+1}-a_j+\varepsilon}{a_{j+1}-a_j}\right)=0, \tag{11}$$

$$u_3\left(1-\frac{a_j-a_{j-1}+\varepsilon+E(z)}{b_j-b_{j-1}}\right)=0, \tag{12}$$

$$u_4\left(1-\frac{a_{j+1}-a_j+\varepsilon+E(z)}{b_{j+1}-b_j}\right)=0, \tag{13}$$

$$u_5\left(a_{j-1}-a_j\right)=0, \tag{14}$$

$$u_6\left(a_j-a_{j+1}\right)=0. \tag{15}$$

Many cases of solving equations (9) - (15) have been found. We found that the case: $u_1=u_2=\cdots=u_6=0$ is the only case that gives optimal values for $\{a_j; j\ge 0\}$. And therefore, the optimal value of a_{j+1} is given by solving this equation,

$$\left[\frac{1}{3}\left(a_j-a_{j-1}+\varepsilon\right)+\frac{2}{3}a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon\right)^{-2}-a_0^2\left(a_j-a_{j-1}+\varepsilon\right)^{-3}\right]\left[\gamma\left(\alpha,b_{j-1}\right)\right]$$

$$+\left[\frac{-1}{3}\left(a_{j+1}-a_j+\varepsilon\right)-\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon\right)^{-2}+a_0^2\left(a_{j+1}-a_j+\varepsilon\right)^{-3}\right]\left[\gamma\left(\alpha,b_j\right)\right]$$

$$+\left[\frac{1}{3}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)+\frac{2}{3}a_0^{3/2}\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^{-2}-a_0^2\left(a_j-a_{j-1}+\varepsilon+E(z)\right)^{-3}\right]\left[\gamma\left(a_{j-1},\beta\right)\right]$$

$$-\left[\frac{-1}{3}\left(a_{j+1}-a_j+\varepsilon+E(z)\right) -\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-2}+a_0^2\left(a_{j+1}-a_j+\varepsilon +E(z)\right)^{-3}\right]\left[\gamma(a_j,\beta)\right]$$

$$+\left[\frac{1}{6}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^2 -\frac{2}{3}a_0^{3/2}\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-1}+\frac{1}{2}a_0^2\left(a_{j+1}-a_j+\varepsilon+E(z)\right)^{-2}\right]f(a_j) =0$$

And by the same manner, the optimal values of b_{j+1} is getting after solving NLP(3).

NLP(3) :

$$\min_{b_j}\left[\frac{\left(b_j-b_{j-1}+\xi-E(z)\right)^4-4b_0^{3/2}\left(b_j-b_{j-1}+\xi-E(z)\right)+3b_0^2}{6\left(b_j-b_{j-1}+\xi-E(z)\right)^2}\right]\left[\gamma(\alpha,b_{j-1})\right]$$

$$+\left[\frac{\left(b_{j+1}-b_j+\xi-E(z)\right)^4-4b_0^{3/2}\left(b_{j+1}-b_j+\xi-E(z)\right)+3b_0^2}{6\left(b_{j+1}-b_j+\xi-E(z)\right)^2}\right]\left[\gamma(\alpha,b_j)\right]$$

$$+\left[\frac{(b_j-b_{j-1}+\xi)^4-4b_0^{3/2}(b_j-b_{j-1}+\xi)+3b_0^2}{6(b_j-b_{j-1}+\xi)^2}\right]\left[\gamma(a_{j-1},\beta)\right] -\left[\frac{(b_{j+1}-b_j+\xi)^4-4b_0^{3/2}(b_{j+1}-b_j+\xi)+3b_0^2}{6(b_{j+1}-b_j+\xi)^2}\right]\left[\gamma(a_j,\beta)\right]$$

subject to:

$$1-\frac{b_j-b_{j-1}+\xi-E(z)}{b_j-b_{j-1}}\le 0,$$

$$1-\frac{b_{j+1}-b_j+\xi-E(z)}{b_{j+1}-b_j}\le 0,$$

$$1-\frac{b_j-b_{j-1}+\xi}{b_j-b_{j-1}}\le 0,$$

$$1-\frac{b_{j+1}-b_j+\xi}{b_{j+1}-b_j}\le 0,$$

$$b_{j-1}-b_j<0,\quad b_j-b_{j+1}<0,\quad -\xi\le 0,$$

The optimal values of b_{j+1} are given after solving this equation,

$$\left[\frac{1}{3}\left(b_j-b_{j-1}+\xi-E(z)\right)+\frac{2}{3}b_0^{3/2}\left(b_j-b_{j-1}+\xi-E(z)\right)^{-2}-b_0^2\left(b_j-b_{j-1}+\xi-E(z)\right)^{-3}\right]\left[\gamma(\alpha,b_{j-1})\right]$$

$$+\left[\frac{-1}{3}\left(b_{j+1}-b_j+\xi-E(z)\right)-\frac{2}{3}b_0^{3/2}\left(b_{j+1}-b_j+\xi-E(z)\right)^{-2}+b_0^2\left(b_{j+1}-b_j+\xi-E(z)\right)^{-3}\right]\left[\gamma(\alpha,b_j)\right]$$

$$+\left[\frac{1}{6}\left(b_{j+1}-b_j+\xi-E(z)\right)^2-\frac{2}{3}b_0^{3/2}\left(b_{j+1}-b_j+\xi-E(z)\right)^{-1}+\frac{1}{2}b_0^2\left(b_{j+1}-b_j+\xi-E(z)\right)^{-2}\right]f(b_j)$$

$$+\left[\frac{1}{3}(b_j-b_{j-1}+\xi)+\frac{2}{3}b_0^{3/2}(b_j-b_{j-1}+\xi)^{-2}-b_0^2(b_j-b_{j-1}+\xi)^{-3}\right]\left[\gamma(a_{j-1},\beta)\right]$$

$$-\left[\frac{-1}{3}(b_{j+1}-b_j+\xi)-\frac{2}{3}b_0^{3/2}(b_{j+1}-b_j+\xi)^{-2}+b_0^2(b_{j+1}-b_j+\xi)^{-3}\right]\left[\gamma(a_j,\beta)\right]=0$$

Conclusion

A coordinated search method to find a target on the line that has been randomly placed and whose initial position has a known probability distribution has been described. The searchers move down the line at random speeds throughout time. We presented the expected value of the target's detection time. We also provide the optimal search strategy for minimizing this expected value. In the future, by taking into account the collection of lines, it appears that the proposed model will be adaptable to the case of multiple searchers to detect a lost target.

Declaration

Ethics approval and consent to participate

This study does not contain any studies with human participants or animals performed by any of the authors.

Consent for publication

the authors have approved the manuscript for submission.

Availability of data and material

No data have been used.

Competing interests

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Author contribution

authors are equally contributed to the draft of the manuscript; they have read and approved the final manuscript.

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البحث التنسيقي المعمم لهدف ساكن بشكل عشوائي

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لقد تمت دراسة طريقة بحث منسقة للعثور على هدف مفقود يقع عشوائيا على الخط بتوزيع احتمالي معروف. يبدأ باحثان البحث من اي نقطة تقع علي الخط المستقيم كلا منهما فى اتجاه ويتحرك الباحثان على طول الخط بسرعات عشوائية طوال الوقت. لقد قدمنا القيمة المتوقعة لزمن اكتشاف الهدف. كما قدمنا استراتيجية البحث المثلى لتقليل هذه القيمة المتوقعة .