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Research Article

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A Comparative Study of Effective Techniques for Solving A new Model of (n+1) Dimensional Fractional Burgers' Equation

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KEY WORDS

Laplace Adomian decomposition method, Caputo fractional derivative, Laplace variational iteration method, Lagrange multiplier, Burgers' equation, Reduced differential transform method.

ABSTRACT

The present work offers a new model of (n+1)-dimensional fractional Burgers' equation ((n+1)D-FBE) and presents a comparative numerical study of three efficient semi analytical techniques for solving the ((n+1)D-FBEs). These techniques include the Laplace Adomian decomposition method (LADM), the Laplace variational iteration method (LVIM) and the reduced differential transform method (RDTM). The suggested approaches consider the use of the suitable initial conditions and find the solutions without any discretization or limiting traditions. Furthermore, their solutions are in the form of quickly convergent series with easily calculable terms. Numerical studies of four numerical applications are provided to certify the effectiveness and reliability of the suggested approaches, also to compare their computational effectiveness with each other and with other supplementary methods in the available literature. In addition to explore the properties of the solutions when changing the fractional derivative parameter. Numerical results demonstrate the effectiveness and accuracy of the suggested methods.

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Introduction

The Burgers' equation is one of the most well-known equations containing both non-linear propagation effects and diffusive effects. Burgers' equation, being a nonlinear PDE, stand for a diverse physical issue resulting in engineering, which are basically hard to solve. Lately, a growing attract has been devised within the scientific society, for studying non-linear convective–diffusion partial differential equations partly due to the tremendous advancement in computational capacity **(e.g. Srivastava et al., 2022; Alqhtani et al., 2022).** Burgers' equation is suitable for the analysis of various significant. It can be seen that Burgers' equation is a basic model of fluid dynamics and is used as a computational device for dealing with more complex phenomena. It may be used to model the airflow around a wing of the aircraft **(Das, 1994)**, acoustic transmission (**Moslem et al., 2008),** heat conduction (**Rashidi and Erfani, 2009)**, fluid flow in a pipe **(Dutta et al., 2016)**, etc. Due to the wide range of uses of Burgers' equation, it is widely studied and various numerical techniques are available. Most of these algorithms belong either to the finite element technique or to the finite difference technique and the spectral technique **(e.g. Khater et al., 2008; INAN and BAHADIR, 2013; Doha et al., 2013; Bhrawy et al., 2015; Yokus, 2018; Ahmed et al., 2020)**. J. M. Burgers (1895-1981) **(e.g. Bateman, 1915; Burgers, 1948; Lax, 1973; Garra, 2011)** stated the mathematical structure of Burgers' equation. The most recent progress has been assumed in the field of fractional differential equations (FDEs) and fractional calculus (FC). Nanotechnology, electromagnetic waves, ion-acoustic waves, bio-informatics, electrode-electrolyte polarisation, viscoelasticity, chemical engineering, mechanical, heat conduction, diffusion, and almost all disciplines of study and technology make extensive use of FDEs (**Garra, 2011; Jiang et al., 2012; Sabatier et al., 2007; Debnath, 2003)**. In comparison to traditional integer-order derivatives, fractional derivatives provide more realistic simulations of real-world issues. Also, it is the same purpose why many fractional models of the Burgers' equation (**Bhrawy et al., 2015; Yokus, 2018; Ahmed, 2020; Jiang et al., 2012; Momani, 2006; Inc, 2008; Saad and Al-Sharif, 2017; Yokus and Kaya, 2017; Sripacharasakullert et al., 2019; Ahmed et al., 2019; Ahmed, 2019; Ahmed, 2020; Kilicman, et al., 2021)** have been suggested and studied lately. The fractional model is obtained by substituting fractional equivalents for the time/space integer derivatives and the initial / boundary conditions. The existence and uniqueness of generalised Burgers' equations (GBEs) solutions governed by multi-parameter fractional derivatives are revealed in (**Jiang et al., 2012)**. The non-perturbation analytical solutions of the fractional GBEs are obtained by the Adomian decomposition method **(Momani, 2006)**. In (**Inc, 2008; Saad and Al-Sharif, 2017**), the variational iteration method is applied to gain the approximate and numerical solutions of the class of time and time- space- fractional Burgers' equations. In **(Yokus and Kaya, 2017)**, the expansion method and Cole-Hopf transformation are used to find the exact solution for the traveling wave equation of nonlinear time

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fractional Burgers' equation. An estimated analytical solution of fractional multi-dimensional Burgers' equation is attained by the homotopy perturbation method **(Sripacharasakullert et al., 2019)**. In **(Ahmed et al., 2019)** the LVIM, LADM and RDTM are successfully applied for solving the one- and two-dimensional fractional coupled Burgers'
equation. While in **(Kilicman et al., 2021)** the homotopy perturbation method is considered to solve the (1+n)-dimensional fractional M-Burger's equation with a force term. Notably, there is a need to improve more efficacious analytical and numerical techniques to address changing models of FBE, especially in multidimensional spaces. The main goal of this paper is to spur the use of the LADM, LVIM, and RDTM to create semi-analytical approximate solutions of the (n+1) dimensional space time-fractional Burgers' equation ((n+1)D-STFBEs) of the form:

$$D_t^{\alpha} w\left(\vec{x},t\right) + \epsilon\ w\left(\vec{x},t\right)\ \sum_{i=0}^{n} D_{x_i}^{\beta} w(x_i,\ t) - \mu \sum_{i=0}^{n} D_{x_i}^{2\beta} w(x_i,\ t)\ = 0. \quad (1)$$

With the initial condition,

$$w\left(\vec{x},0\right) = f_0\left(\vec{x}\right). \quad (2)$$

Where $0 < t \leq T,\ \vec{x} = (x_1, x_2, \cdots, x_n),\ 0 < \alpha, \beta \leq 1, D_t^{\alpha}, D_{x_i}^{\beta}, D_{x_i}^{2\beta}$ are the Caputo fractional derivatives of orders α, β and 2β, respectively, μ is the viscosity coefficie,ϵ is the coefficient of nonlinear convection term $w\left(\vec{x},t\right)\ \sum_{i=0}^{n} D_{x_i}^{\beta} w$,$D_t^{\alpha} w\left(\vec{x},\ t\right)$ is unsteady term, $w\left(\vec{x},t\right)\ \sum_{i=0}^{n} D_{x_i}^{\beta} w(x_i,\ t)$ is the nonlinear convection term, $D_{x_i}^{2\beta} w(x_i,\ t)$ is the diffusion term. In this article, we utilize LVIM, LADM and RDTM to the (n+1) D-TSFBEs (1). The suggested methods are vigorous and effective in ruling suitable solutions for extensive forms of linear and nonlinear differential equations. They Also consider the use of the suitable initial conditions and find the solutions without any discretization perturbation, or any other restrictive assumptions that may change the physical behavior of the model under study. Furthermore, their solutions are in the form of quickly convergent series with easily calculable terms. The LADM is an ingenious combination of the Laplace transform and Adomain decomposition method (ADM) **(Khuri, 2001; Wazwaz, 2010; Jafari et al., 2013; Ahmed et al., 2017)**, and was introduced by **(Khuri, 2001)**. The LVIM **(Guo-Cheng and Dumitru Baleanu, 2013)** is a new reform of the variational iteration method (VIM) which is excellently used to solve extensive classes of differential equations (**Ahmed et al., 2019; Ahmed, 2019; Ahmed, 2020; Ahmed et al., 2017; Guo-Cheng and Dumitru Baleanu, 2013)**. **Zhhou, (1986)** presented a simplified differential transformation that he used to nonlinear and linear initial value circuit problems. The Taylor series expansion influences RDTM, which is a semi-analytical approach. It differs from the widely used high-order Taylor's series approach. The vital derivatives of the data functions must be symbolically scheming. An iterative technique is used by RDTM to obtain a polynomial series solution. RDTM **(Ahmed et al., 2019; Zhhou, 1986; Sohail and**

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Mohyud-Din, 2012) was effectively used to produce analytical approximate solutions to many kinds of FDEs. In order to demonstrate the effectiveness of the proposed methods, four numerical applications are provided. Graphic illustrations of the proposed three techniques with the exact solutions are presented. To explain the efficacy and precision of the proposed methodologies, numerical comparisons with other published results in the current literature are provided. The outcomes show that the proposed procedures are very successful, rigorous, and straightforward. The contents of the present article are as follows:

The section "Preliminaries and definitions" contains basic definitions and mathematical prefaces for fractional Caputo derivatives, which are necessary for constructing numerical solutions. The expounding of LVIM, LADM and RDTM on the (n+1) D-STFBEs are exposed in "Methods" section. The numerical and graphical results of four different applications are described in "Results and discussion" section. We provide conclusions that conclude this paper in “Conclusion“ section.

Preliminaries and Definitions

Definition 1: Suppose that: $\alpha > 0, \quad x > a, \quad \alpha, \quad a \in \mathbb{R}$. Then the fractional operator

$$D_c^{\alpha} w(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} w(\tau)\, d\tau, & m-1 < \alpha < m,\ m \in \mathbb{N},\ x > a, \\ \frac{d^m}{dx^m} w(x), & \alpha = m,\ m \in \mathbb{N},\ x > a, \end{cases} \tag{3}$$

is known as the Caputo fractional derivative (CFD) operator (**Miller and Ross, 1993; Li et al., 2011)** of order α.This operator is introduced by Italian mathematician Caputo in 1967.

Definition 2: Suppose $W(s)$ is the Laplace transform of $w(x)$. Then the Laplace transform of the CFD (3) is defined as

$$\left\{D_C^{\alpha}\left(w(x)\right); s\right\} = s^{\alpha} W(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} w^{(k)}(0), \qquad m-1 < \alpha < m. \tag{4}$$

Definition 3: The inverse differential transform of $W_k(x)$ is calculated as follows:

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x) t^{k\alpha}. \tag{5}$$

Definition 4: The analytical and continuously differentiable function $w(x,t)$ w.r.t. t, the t-dimensional transformed function $W_k(x)$ is provided by

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} [(D_t^{\alpha})^k w(x,t)]_{t=0}, \tag{6}$$

where $0 < \alpha \leq 1, (D_t^{\alpha})^k = D_t^{\alpha}.D_t^{\alpha}.D_t^{\alpha}..D_t^{\alpha}$, k-times and D_t^{α} is the fractional differential operator with respect to time of order α.

An Study of The Semi-Analytical Methods

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The Application Process of LADM to the (1+ n)D-STFBE

Consider the (n+1)D- STFBE represented by Eq.(1). The procedure solution of using LADM for solving Eq.(1) is as follows:

***Step1*:** Implementing the Laplace transform of Eq.(1) and utilizing the initial conditions (2), we get

$$£_t[w(\vec{x},t)] = \frac{1}{s} f_0(\vec{x}) + \frac{1}{s^\alpha} £_t\left[-\epsilon w(\vec{x},t)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w(x_i\,,t)\right) + \mu \sum_{i=0}^{n} D_{x_i}^{2\beta} w(x_i\,,t)\right]. \quad (7)$$

Step2: LADM describes the solutions *u*(*x*, *t*) by the infinite series:

$$w(\vec{x},t) = \sum_{n=0}^{\infty} w_n(\vec{x},t). \quad (8)$$

The nonlinear parts $w\left(\vec{x},t\right) \ \sum_{i=0}^{n} D_{x_i}^{\beta} w(x_i,\ t)$ in (7) are decomposed as:

$$\left(\sum_{n=0}^{\infty}\sum_{i=0}^{n} D_{x_i}^{\beta} w_n(x_i\,,t)\right)\sum_{n=0}^{\infty} w_n(\vec{x},t) = \sum_{n=0}^{\infty} A_n, \quad (9)$$

where A_n is the Adomian polynomials. The general formula for the Adomian polynomials is

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty} \lambda^k \left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_k(x_i,\ t)\right)\right)\sum_{k=0}^{\infty} \lambda^k w_k\left(\vec{x},t\right)\right]. \quad (10)$$

The first few components of A_n are

$$\begin{aligned} A_0 &= w_0\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_0(x_i,t)\right),\\ A_1 &= w_0\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_1(x_i,t)\right) + w_1\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_0(x_i,\ t)\right), \quad (11)\\ A_2 &= w_0\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_2(x_i,t)\right) + w_1\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_1(x_i,t)\right)\\ &\quad + w_2\left(\vec{x},t\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_0(x_i,t)\right), \ldots \end{aligned}$$

Step3: Following the decomposition analysis strategy, Eq. (7) is converted into a set of recursive equations given by

$$£_t[w_0(\vec{x},t)] = \frac{1}{s}\left(f_0(\vec{x})\right),$$

$$£_t[w_1(\vec{x},t)] = \frac{1}{s^\alpha} £_t\left[-\epsilon A_0 + \mu \sum_{i=0}^{n} D_{x_i}^{2\beta} w_0(x_i\,,t)\right], \ldots..$$

Then,

$$£_t[w_{k+1}(\vec{x},t)] = \frac{1}{s^\alpha} £_t\left[-\epsilon A_k + \ \mu \sum_{i=0}^{n} D_{x_i}^{2\beta} w_k(x_i\,,t)\right]. \quad (12)$$

Step4: Implementing the inverse Laplace transform, the components $w_k,\ k \geq 1$ are given as

$$\begin{aligned} w_0(\vec{x},t) &= £_t^{-1}\left(\frac{1}{s}\left(f_0(\vec{x})\right)\right) = f_0(\vec{x}),\\ w_1(\vec{x},t) &= f_1(\vec{x})\frac{t^\alpha}{\Gamma(1\alpha+1)}, \quad (13)\\ w_2(\vec{x},t) &= f_2(\vec{x})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \ ,\ldots.. \end{aligned}$$

So the N Approximate solution is presented by:

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$$\phi_N\left(\vec{x},t\right)=\sum_{i=0}^{N} w_i\left(\vec{x},t\right)=\sum_{i=0}^{N} f_i\left(\vec{x}\right)\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \tag{14}$$

Where

$$f_0\left(\vec{x}\right)=w\left(\vec{x},0\right)$$

$$f_1\left(\vec{x}\right)=-\epsilon\ f_0\left(\vec{x}\right)\ \left(\sum_{i=0}^{n} D_{x_i}^{\beta} f_0(x_i\ )\right)+\mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} f_0(x_i\ )\right), \tag{15}$$

$$f_2\left(\vec{x}\right)=-\epsilon\ f_0\left(\vec{x}\right)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} f_1(x_i\ )\right)-\epsilon\ f_1(x)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} f_0(x_i\ )\right)+\mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} f_1(x_i\ )\right).$$

And the exact solution will be $\lim_{N\to\infty}\phi_N\ \left(\vec{x},t\right)$.

In some cases the exact solution in the closed form may also establish. Theoretical studies on the convergence of ADM are debated in **(Adomian and Rach, 1992; Abbaoui and Cherruault, 1995; Cherruault et al., 1995; Babolian and Biazaar, 2002; El-Kalla, 2007; Ray, 2014)**. Theorem 1 **(Ray, 2014)** establishes sufficient conditions for the existence of a unique solution. Theorem 2 **(Ray, 2014)** demonstrates the convergence of the form of the series solution realized by ADM. The maximum errors of the solution that is gained by the ADM are proven in **(Ray, 2014)**.

The Application Process of LVIM to the (1+ n)D-STFBE:

Consider the (1+n) D-STFBE represented by Eq.(1). The following steps describe the procedure of the LVIM

Step1*:* Employing the Laplace transform to Eq.(1), we obtain

$$s^{\alpha}£_t[w(\vec{x},t)]-\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k w(\vec{x},t)}{\partial t^k}\bigg|_{t=0}=-\epsilon £_t\left[w(\vec{x},t)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w(x_i\ ,t)\right)\right]$$
$$+\mu\ £_t\left[\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} w(x_i\ ,t)\right)\right]\quad . \tag{16}$$

***Step2*:** Describe the vital iterative structure including the Lagrange multiplier as

$$£_t[w_{N+1}(\vec{x},t)]=£_t[w_N(\vec{x},t)]+\lambda(s)[s^{\alpha}£_t[w_N(\vec{x},t)]-\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k w(\vec{x},t)}{\partial t^k}\bigg|_{t=0}$$
$$-\epsilon £_t\left[w_N(\vec{x},t)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_N(x_i\ ,t)\right)\right]+\mu\ £_t\left[\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} w_N(x_i\ ,t)\right)\right]. \tag{17}$$

Considering $£_t\left[w_N(\vec{x},t)\left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_N(x_i\ ,t)\right)\right]$ and $£_t\left[\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} w_N(x_i\ ,t)\right)\right]$ as as restricted terms, so the Lagrange multiplier can be derived as

$$\lambda(s)=\frac{-1}{s^{\alpha}}\ . \tag{18}$$

***Step3*:** By using Eq.(18) with the inverse Laplace transform $£_t^{-1}$, the iteration formula Eq.(17) is obtained as

$$w_{N+1}(\vec{x},t)=£_t^{-1}\{\frac{1}{s^{\alpha}}\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k w(\vec{x},t)}{\partial t^k}\bigg|_{t=0}+\frac{\epsilon}{s^{\alpha}}£_t\left[w_N(\vec{x},t)\ \left(\sum_{i=0}^{n} D_{x_i}^{\beta} w_N(x_i\ ,t)\right)\right]$$

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$$-\frac{\mu}{s^{\alpha}}£_t\Big[\sum_{i=0}^{n} D_{x_i}^{2\beta} w_N(x_i\,,t)\Big]\}. \tag{19}$$

Step 4: The initial iteration $w_0\left(\vec{x},t\right)$ is assumed as

$$w_0(\vec{x},t) = £_t^{-1}[\frac{1}{s^{\alpha}}\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k w(\vec{x},t)}{\partial t^k}\Big|_{t=0}]. \tag{20}$$

From Eqs.(2), (20) and (19), we get the solution as follows:

$$w_0(\vec{x},t) = £_t^{-1}\left[\frac{1}{s^{\alpha}}\sum_{k=0}^{m-1} s^{\alpha-k-1}\frac{\partial^k w(\vec{x},t)}{\partial t^k}\Big|_{t=0}\right] = w(\vec{x},0) = f_0(\vec{x}),$$

$$w_1(\vec{x},t) = f_0(\vec{x}) + q_1(\vec{x})\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{21}$$

$$w_2(\vec{x},t) = f_0(\vec{x}) + q_1(\vec{x})\frac{t^{\alpha}}{\Gamma(\alpha+1)} + q_2(\vec{x})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + q_3(\vec{x})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$\vdots$

Where

$$f_0(\vec{x}) = w(\vec{x},0),$$

$$q_1(\vec{x}) = \mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} f_0(x_i\,)\right) - \epsilon f_0(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} f_0(x_i\,)\right),$$

$$q_2(\vec{x}) = \mu\Big(\sum_{i=0}^{n} D_{x_i}^{2\beta} q_1(x_i\,)\Big) - \epsilon q_1(\vec{x})\Big(\sum_{i=0}^{n} D_{x_i}^{\beta} f_0(x_i\,)\Big) - \epsilon f_0(\vec{x})\Big(\sum_{i=0}^{n} D_{x_i}^{\beta} q_1(x_i\,)\Big), \tag{22}$$

$$q_3(\vec{x}) = -\epsilon\, q_1(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} q_1(x_i\,)\right)\frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2}.$$

The exact solution is $w\left(\vec{x},t\right) = lim_{N\to\infty} w_N(\vec{x},t)$.

The convergence of VIM was discussed in detail in theorem 1 and theorem 2 **(Obibat, 2010; Zedan et al., 2014)**. Theorem 3 **(Obibat, 2010; Zedan et al., 2014)** defines the maximum errors of the VIM solution.

The Application Process of RDTM to the (1+ n)D-STFBE

Consider the (1+n) D-STFBE represented by Eq.(1). We can summarize the procedure of the solution of the RDTM in the following steps:

Step1*:* Employing the RDTM to Eq. (1), the subsequent repetitive formula is attained:

$$W_{k+1}(\vec{x}) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[\mu\Big(\sum_{i=0}^{n} D_{x_i}^{2\beta} W_k(x_i\,)\Big) - \epsilon\sum_{r=0}^{k} W_r(\vec{x})\Big(\sum_{i=0}^{n} D_{x_i}^{\beta} W_{k-r}(x_i\,)\Big)\right], \tag{23}$$

and the transformed initial condition is

$$W_0\left(\vec{x}\right) = w\left(\vec{x},0\right) = r_0\left(\vec{x}\right). \tag{24}$$

Step 2*:* Using the initial iteration (24), the following $W_k\left(\vec{x}\right)$ values are obtained

$$W_1(\vec{x}) = \frac{1}{\Gamma(\alpha+1)}\left[\mu\Big(\sum_{i=0}^{n} D_{x_i}^{2\beta} W_0(x_i\,)\Big) - \epsilon W_0(\vec{x})\Big(\sum_{i=0}^{n} D_{x_i}^{\beta} W_0(x_i\,)\Big)\right],$$

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which can be written as

$$W_1(\vec{x}) = \frac{1}{\Gamma(\alpha+1)} r_1(\vec{x}). \tag{25}$$

$$W_2(\vec{x}) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[(\mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} W_1(x_i)\right) - \epsilon W_0(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} W_1(x_i)\right) - \epsilon W_1(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} W_0(x_i)\right)\right],$$

which can be written as

$$W_2(\vec{x}) = \frac{1}{\Gamma(2\alpha+1)} r_2(\vec{x}),$$

$$\vdots$$

Step3: By means of the inverse transformation of the set of values; $\left\{W_k\left(\vec{x}\right)\right\}_{k=0}^{\infty}$, the approximate solution is given as,

$$\widetilde{w}_N(\vec{x},t) = \sum_{k=0}^{N} W_k(\vec{x}) t^{k\alpha},$$

$$\widetilde{w}_N(\vec{x},t) = r_0(\vec{x}) + r_1(\vec{x})\frac{t^{\alpha}}{\Gamma(\alpha+1)} + r_2(\vec{x})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \tag{26}$$

where N is order of the estimate solution and

$$r_0(\vec{x}) = w(\vec{x},0),$$

$$r_1(\vec{x}) = \mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} r_0(x_i)\right) - \epsilon r_0(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} r_0(x_i)\right), \tag{27}$$

$$r_2(\vec{x}) = \mu\left(\sum_{i=0}^{n} D_{x_i}^{2\beta} r_1(x_i)\right) - \epsilon r_0(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} r_1(x_i)\right) - \epsilon r_1(\vec{x})\left(\sum_{i=0}^{n} D_{x_i}^{\beta} r_0(x_i)\right), \ldots$$

Hence, the exact solution is assumed to be $w\left(\vec{x},t\right) = \lim_{N\to\infty}\widetilde{w}_N\left(\vec{x},t\right)$.The convergence of the DTM was discussed in **(Odibatet et al., 2016)** while the maximum absolute error of RDTM was given in **(Odibatet al., 2016)**.

Numerical Applications and Discussion

To illustrate the relevance and efficacy of the approaches proposed, We solved four applications using the approaches described in this research. Applications 1 and 2 represent the one-dimensional fractional Burger equation (FBE), whereas Applications 3 and 4 represent the two-dimensional fractional Burger equation (FBE)

Application 1 (Sripacharasakullert et al., 2019): Now, let's test 1D-TSFBE

$$D_t^{\alpha} w(x,t) + \epsilon\ w(x,t)\ D_x^{\beta} w(x,t) - \ D_x^{2\beta} w(x,t) = 0. \tag{28}$$

With initial condition

$$w(x,0) = x, \tag{29}$$

where $0 < t \leq 1,\ 0\ \leq x \leq 1,\ 0 < \alpha,\beta \leq 1.$

At the special case $\alpha = \beta = 1$, the exact solution of (28) is

$$w(x,t) = \frac{x}{1+\epsilon\ t}\,. \tag{30}$$

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LADM Solution for Application 1

By using the initial conditions (29), Eqs. (15) takes the following form

$$f_0(x) = x,$$
$$f_1(x) = \frac{\epsilon\ x^{2-\beta}}{\Gamma(2-\beta)}, \tag{31}$$
$$f_2(x) = \frac{\epsilon^2\ x^{3-2\beta}\ \Gamma(3-\beta)}{\Gamma(3-2\beta)\Gamma(2-\beta)} + \frac{\epsilon^2\ x^{3-2\beta}}{(\Gamma(2-\beta))^2},$$

by combining Eqs. (29), (31) and (14), the solution series is set as

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots.$$
$$= f_0(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \tag{32}$$

LVIM Solution for Application 1

By using the initial condition (29) within Eqs.(21), the first few iteration of LVIM solution are:

$$w_0(x) = x,$$
$$w_1(x) = x - \left(\frac{\epsilon x^{2-\beta}}{\Gamma(2-\beta)}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{33}$$
$$w_2(x) = x - \left(\frac{\epsilon x^{2-\beta}}{\Gamma(2-\beta)}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{\epsilon^2 x^{3-2\beta}(\Gamma[3-2\beta]+\Gamma[2-\beta]\Gamma[3-\beta])}{\Gamma(3-2\beta)\left(\Gamma(2-\beta)\right)^2}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\left(\frac{\epsilon^2 x^{4-3\beta}\ \Gamma(3-\beta)\Gamma(1+2\alpha)}{\left(\Gamma(2-\beta)\right)^2\Gamma(3-2\beta)\left(\Gamma(1+\alpha)\right)^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

RDTM Solution for Application 1

By sing Eq. (29), then Eqs. (25) take the sequential form

$$W_0(x) = x,$$
$$W_1(x) = \frac{-1}{\Gamma(\alpha+1)}\left(\frac{\epsilon x^{2-\beta}}{\Gamma(2-\beta)}\right),$$
$$W_2(x) = \frac{1}{\Gamma(2\alpha+1)}\left(\frac{\epsilon^2 x^{3-2\beta}\ \Gamma(3-\beta)}{\Gamma(3-2\beta)\Gamma(2-\beta)} + \frac{\epsilon^2 x^{3-2\beta}}{\left(\Gamma(2-\beta)\right)^2}\right),$$

and so on, so the series solution of Eq.(26) will be

$$w(x,t) = x + \left(-\frac{\gamma x^{2-\beta}}{\Gamma(2-\beta)}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\left(\frac{\epsilon^2 x^{3-2\beta}\ \Gamma(3-\beta)}{\Gamma(3-2\beta)\Gamma(2-\beta)} + \frac{\epsilon^2 x^{3-2\beta}}{\left(\Gamma(2-\beta)\right)^2} - \frac{\epsilon^2 x^{2-\beta}\ \Gamma(1+\alpha)}{\Gamma(2-\beta)}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \tag{34}$$

Numerical Study of Application 1

The numerical results of application 1 calculated by using five terms of the LADM, RDTM and LVIM are tabulated in Table 1 and graphically illustrated in Figuers 1-3. Table 1 lists the absolute errors (AEs) for the three proposed methods at $\alpha = \beta = 1,\ t = 0.02,\ \epsilon = 1$ and method in **(Yokus, 2018)** at different values of x. The tabulated data show that the suggested three methods are more effective than the results in **(Yokus, 2018)** Figure 1((a), (b) and (c)) clarifies the space-time surfaces of the exact solution w_{approx}

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and the approximate solutions w_{approx} obtained by LADM, LVIM and RDTM, respectively. Figures 2 illustrate the x-direction curves of AEs for the analytical approximate solutions; w_{LADM}, w_{LVIM} *and* w_{RDTM} when $\beta = \alpha = 1,\ \epsilon = 1,\ 0 \leq x \leq 1$ at various values of $t = 0.02,\ 0.1,\ 0.2$ and 0.4. Figures 3 and 4 clarify the behaviors of the estimated solutions of $w_{approx,N}$ by exhausting the suggested three methods at various values of $\alpha = 1$ and $\beta = 0.4,\ 0.6,\ 0.8,\ 1$ and for distinct values of the series solution terms $N = 0,1,2,3,4$ respectively. These Figures reveal the effectively, simplicity and higher accuracy of the proposed techniques.

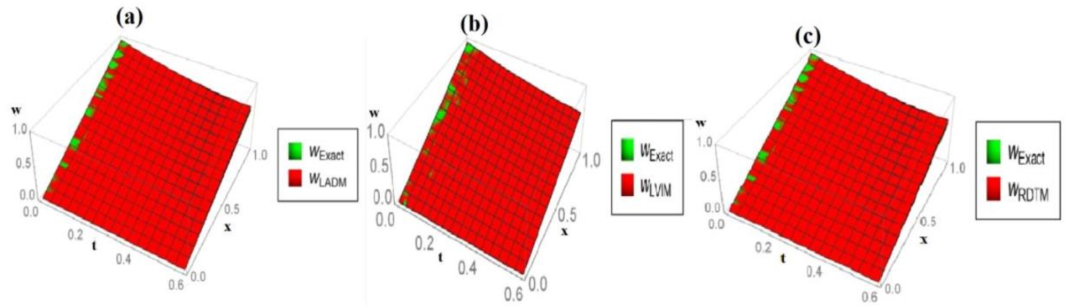

Fig. 1: The space time surfaces of the exact solution w_{Exact} and (a) w_{LADM}, (b) w_{LVIM} (c) w_{RDTM} when $\epsilon = 1, 0 \leq x \leq 1, 0 \leq t \leq 0.6$, $\beta = \alpha = 1$ for application 1

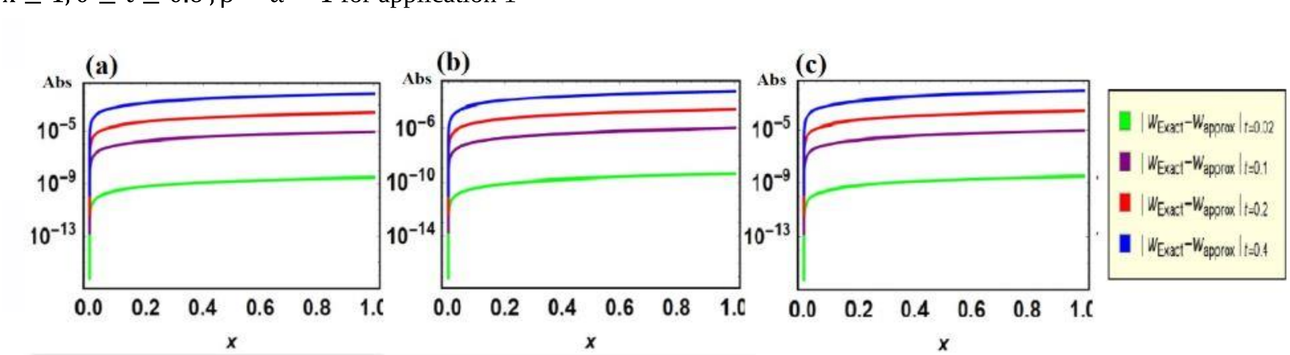

Fig. 2: The $\boldsymbol{x}$-direction of AEs for LADM, LVIM and RDTM respectively, at $\alpha = \beta = 1,\ \epsilon = 1,\ 0 \leq x \leq 1$, and distinct values of $t = 0.02,\ 0.1,\ 0.2,\ 0.4$ for application 1.

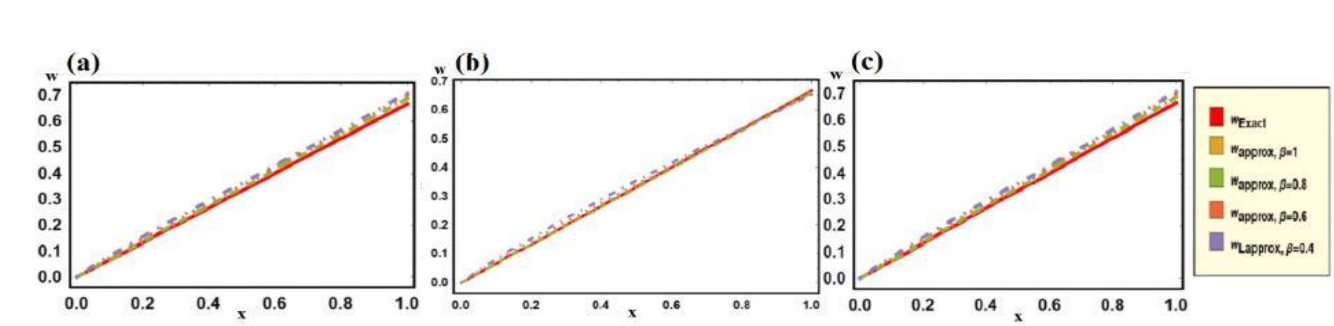

Fig. 3: The x -direction curves of w_{approx} gained by LADM, LVIM and RDTM respectively, for $\epsilon = 1,\ 0 \leq x \leq 1$, $t = 0.5,\ \alpha = 1,\ \beta = 0.4,\ 0.6,\ 0.8, 1$ and exact solution for application 1.

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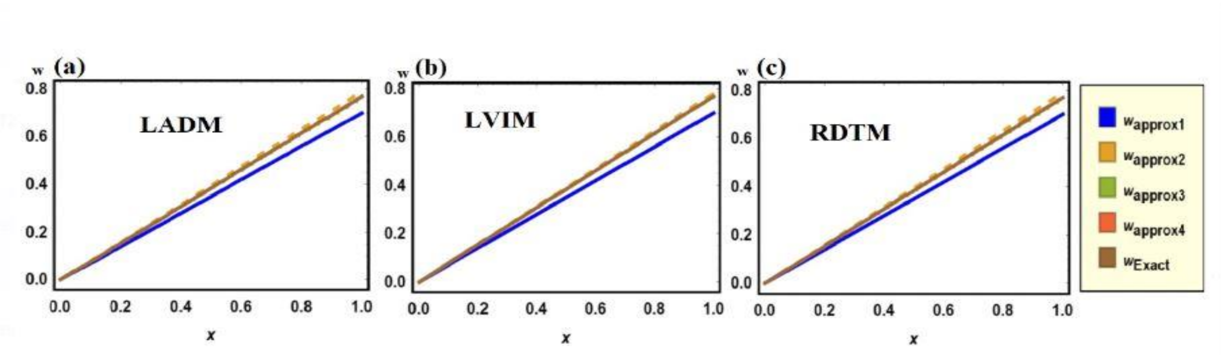

Fig. 4: The x-direction curves of different N of terms of $w_{approx,N}(x,t)$ attained by LADM, LVIM and RDTM respectively, $\epsilon = 1,\ 0 \le x \le 1,\ t = 0.25,\ \alpha = \beta = 1.\ N = 1,\ 2,\ 3,\ 4$ and exact solution for application 1.

Table 1: Comparison of AEs, Obtained by Our Methods and The Results of (Yokus, 2018) when $\alpha = \beta = 1,\ t = 0.02,\ \epsilon = 1$ for application 1.

x	LADM	LVIM	RDTM	FDM (Yokus, 2018)	GTSM (Yokus, 2018)
0.00	0.0	0.0	0.0	0.0	0.0
0.02	6.3 E-11	8.1 E-12	6.3 E-11	1.7 E-04	2.2 E-04
0.04	1.3 E-10	1.6 E-11	1.3 E-10	3.0 E-04	4.4 E-04
0.06	1.9 E-10	2.4 E-11	1.9 E-10	4.0 E-04	6.7 E-04
0.08	2.5 E-10	3.2 E-11	2.5 E-10	4.9 E-04	8.9 E-04
0.10	3.1 E-10	4.0 E-11	3.1 E-10	5.5 E-04	1.1 E-03
0.12	3.8 E-10	4.7 E-11	3.8 E-10	6.0 E-04	1.3 E-03

Application 2: Consider the subsequent 1D- time fractional Burger equation

$$D_t^{\alpha} w(x,t) + \epsilon \frac{\partial w(x,t)}{\partial x} w(x,t) = \mu \frac{\partial^2 w(x,t)}{\partial x^2}, \tag{35}$$

with initial condition

$$w(x,t) = \frac{c}{\epsilon} - \frac{c}{\epsilon}\ \tanh\left[\frac{c}{2\mu}(x)\right], \tag{36}$$

At $\alpha = 1$, the exact solution **(Doha et al., 2013)** is

$$w(x,t) = \frac{c}{\epsilon} - \frac{c}{\epsilon}\ \tanh\left[\frac{c}{2\mu}(x - ct)\right]. \tag{37}$$

LADM Solution for Application 2

By using the initial conditions (37) with the Eqs. (15), we get

$$\begin{aligned} f_0 &= \frac{c}{\epsilon} - \frac{c}{\epsilon}\ \tanh\left[\frac{c}{2\mu}(x)\right], \\ f_1(x) &= \frac{c^3 (\operatorname{sech}\left[\frac{cx}{2\mu}\right])^2}{2\epsilon\ \mu}, \\ f_2 &= \frac{4c^5 (\operatorname{csch}\left[\frac{cx}{\mu}\right])^3 (\sinh\left[\frac{cx}{2\mu}\right])^4}{\epsilon\ \mu^2}. \end{aligned} \tag{38}$$

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The rest of the terms can be calculated in the same way; the solution in series form is provided by.

$$w(x,t) = \frac{c}{\epsilon} - \frac{c}{\epsilon}\tanh\left[\frac{c}{2\mu}(x)\right] + \left(\frac{c^3 \operatorname{sech}\left[\frac{cx}{2\mu}\right]^2}{2\epsilon\mu}\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{4c^5 \operatorname{csch}\left[\frac{cx}{\mu}\right]^3 \sinh\left[\frac{cx}{2\mu}\right]^4}{\epsilon\mu^2}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \quad (39)$$

LVIM Solution for Application 2

By using the initial conditions (36) within Eqs. (21), the first few iterations of LVIM solution are:

$$w_0(x) = \frac{c}{\epsilon} - \frac{c\ \tanh\left[\frac{cx}{2\mu}\right]}{\epsilon},$$

$$w_1 = \frac{c}{\epsilon} - \frac{c\ \tanh\left[\frac{cx}{2\mu}\right]}{\epsilon} + \left[\frac{c^3(\operatorname{sech}\left[\frac{cx}{2\mu}\right])^2}{2\epsilon\ \mu}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (40)$$

$$\mathrm{w}_2 = \frac{c}{\epsilon} - \frac{c\ \tanh\left[\frac{cx}{2\mu}\right]}{\epsilon} + \left[\frac{\left(c^3(\operatorname{sech}\left[\frac{cx}{2\mu}\right])^2\right)}{2\epsilon\ \mu}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$+ \left[\frac{4c^5(\operatorname{csch}\left[\frac{cx}{\mu}\right])^3(\sinh\left[\frac{cx}{2\mu}\right])^4}{\epsilon\ \mu^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left[\frac{8c^7\Gamma[1+2\alpha](\operatorname{csch}\left[\frac{cx}{\mu}\right])^5(\sinh[\frac{cx}{2\mu}])^6}{\epsilon\ \mu^3(\Gamma(1+\alpha))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

RDTM Solution of Application 2

Using Eq. (36), then Eqs. (25) take the form

$$W_0(x) = \frac{c}{\epsilon} - \frac{c}{\epsilon}\ \tanh\left[\frac{c}{2\mu}(x)\right],$$

$$W_1(x) = \frac{0.5c^3 \operatorname{sech}[\frac{cx}{2\mu}]^2}{\epsilon\ \mu\Gamma(1+\alpha)},$$

$$W_2(x) = \frac{4.\,c^5(\operatorname{csch}\left[\frac{c\ x}{\mu}\right])^3(\sinh[\frac{c\ x}{2\mu}])^4}{\epsilon\ \mu^2\Gamma([1+2\alpha)},$$

$\vdots$

In the similar manner, the remainder terms of the solution series can be calculated. So the series solution Eq. (26) takes the following form

$$w(x,t) = \frac{c}{\epsilon} - \frac{c}{\epsilon}\tanh\left[\frac{c}{2\mu}(x)\right] + \left[\frac{0.5\ c^3\ \operatorname{sech}[\frac{cx}{2\mu}]^2}{\epsilon\ \mu}\right]\frac{t^\alpha}{\Gamma(\alpha+1)} + \left[\frac{0.5c^5 \operatorname{sech}[\frac{cx}{2\mu}]^2 \tanh[\frac{cx}{2\mu}]}{\epsilon\ \mu^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \quad (41)$$

Numerical Study of Application 2

The numerical results of application 2 are recorded in Table 2 and clearly shown through Figures 5-8. The numerical comparisons based on the maximum absolute errors (MAEs) are tabulated in Table 2 between the analytical solutions $w_{approx,N}(x,t)$ obtained by the three proposed methods (LADM, LVIM and RDTM) and the Jacobi spectral collocation method (JSCM) in **(Doha et al., 2013)** at $\alpha = 1,\ 0 \le x \le 1,\ 0 < t \le 1,\ \mu = 0.1,\ \epsilon = 10,$ $c = 0.1$ for various choices of the used terms N. It is observed that the analytical solutions attained by the three suggested techniques converge to the exact solution by using few terms of $w_{approx,N}(x,t)$. Also, the

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comparison manifests that the proposed methods are better than JSCM **(Doha et al., 2013)**. To check the effectiveness of the anticipated techniques for different values of the viscosity coefficient, the surface plots of the exact and approximate solutions are described in Figures 5 and 6. The t –direction curves of the exact and the computed solutions of w_{LADM}, w_{LVIM} and w_{RDTM} at different values of x $(x = 0.1,\ 0.4,\ 0.7)$ are depicted in Figure 7. The behaviors of the estimated solutions w_{LADM}, w_{LVIM} and w_{RDTM} at different values of α, $(\alpha = 1,\ 0.95,\ 0.85,\ 0.75\ and\ 0.65)$ with exact solution are displayed in Figure 8. These figures show that the solutions obtained by the five terms of the series solution of the LADM, LVIM and RDTM are in a good agreement with the exact solution. Also, it is clear that by calculating additional terms of the series Eqs. (39), (40) and (41) the achieved error will be smaller

Table 2: Comparison of MAEs for Various Values of N Terms of The Proposed Methods with JSCM (Doha, 2013) for Application 2.

The Proposed methods				**JSCM (Doha et al., 2013) at various choices(α, β)**				
N	**LADM**	**LVIM**	**RDTM**	N	$(0,0)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(-\frac{1}{2},-\frac{1}{2}\right)$	$\left(-\frac{1}{2},\frac{1}{2}\right)$
1	$9.0E-6$	$9.0E-6$	$9.0E-6$	**4**	$1.5E-6$	$2.5E-6$	$6.5E-6$	$3.8E-6$
2	$4.2E-7$	$5.1E-7$	$4.2E-7$					
3	$1.1E-8$	$2.7E-8$	$1.5E-8$	**16**	$1.4E-9$	$1.6E-9$	$6.7E-10$	$6.7E-10$
4	$4.0E-10$	$1.3\ E-10$	$4.0\ E-10$					

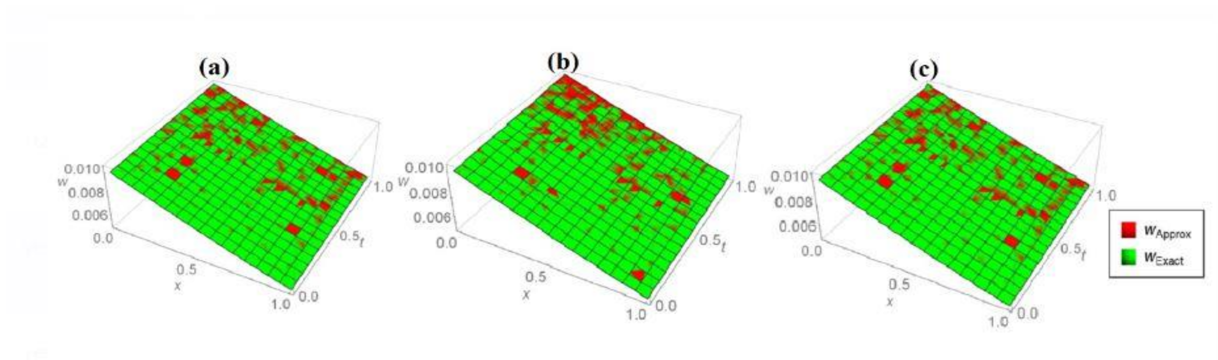

Fig. 5: The space-time surfaces of the approximate solutions w_{approx} of application 2 for $\epsilon = 10,\ c = 0.1\ \mu = 0.1$ achieved by LADM, LVIM and RDTM respectively, with the exact solution.

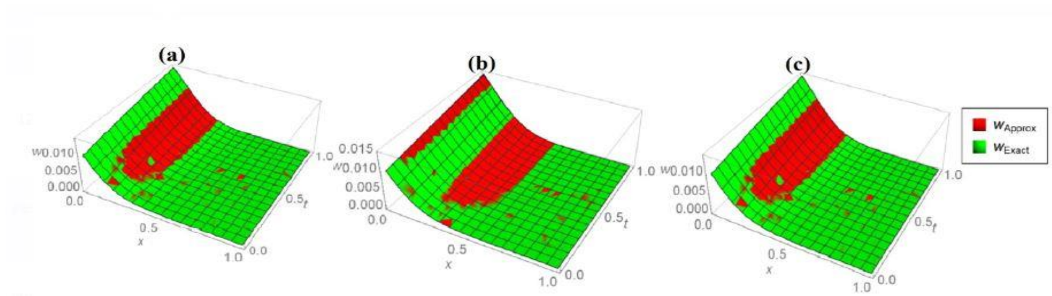

Fig. 6: The space- time surfaces of the approximate solutions w_{approx} of Application 2 for $\epsilon = 10,\ c = 0.1,\ \mu = 0.01$ achieved by LADM, LVIM and RDTM respectively, with the exact solution.

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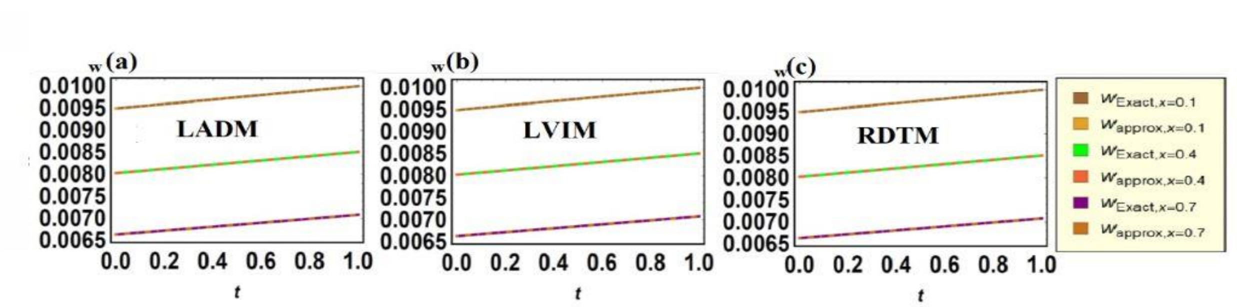

Fig. 7: The t-direction curves of w_{approx} of application 2 gotten by LADM, LVIM and RDTM respectively when $\epsilon = 10,\ c = 0.1,\ \ \mu = 0.1$ and the exact solution.

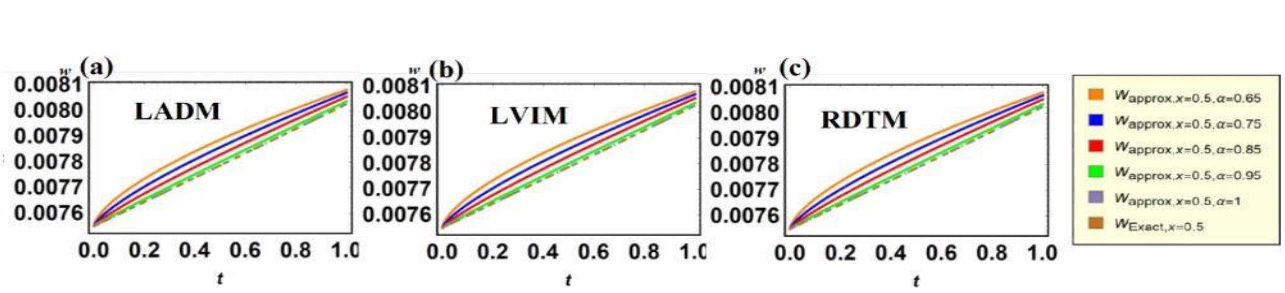

Fig. 8: The $\boldsymbol{t}$-direction of curves of w_{approx} for application 2 attained by LADM, LVIM and RDTM respectively, at different values of α for $\epsilon = 10,\ c = 0.1,\ \mu = 0.1$.

Application 3 (**Sripacharasakullert et al., 2019**): For the following 2D-STFBE:

$$D_t^{\alpha} w(x,y,t) = -\epsilon\ w\left(\ D_x^{\beta} w + D_y^{\beta} w\right) + \mu\left(\ D_x^{2\beta} w + D_y^{2\beta} w\right), \tag{42}$$

with initial condition

$$w(x,y,0) = x + y, \qquad \forall (x,y) \in [0,1] \times [0,1]. \tag{43}$$

The exact solution of Eq. (44) at $\alpha = \beta = 1$ is

$$w(x,y,t) = \frac{x+y}{1+\epsilon\ t}. \tag{44}$$

LADM Solution of Application 3

By using the initial condition (43), Eqs.(15) takes the next form

$$f_0(x,y) = x + y,$$

$$f_1(x,y) = \frac{-\epsilon\ x^{-\beta} y^{-\beta} (x^{1+\beta} y + x^{\beta} y^2 - x^2 y^{\beta} + x y^{1+\beta})}{\Gamma(2-\beta)}, \tag{45}$$

$$f_2(x,y) = \frac{\epsilon^2\ \Gamma(3-\beta)(x+y)\left(x^{2-2\beta} + y^{2-2\beta}\right)}{\Gamma(1+2\alpha)\Gamma(3-2\beta)\Gamma(2-\beta)} + \frac{\epsilon^2\ (x+y)\left(x^{1-\beta} + y^{1-\beta}\right)^2}{\Gamma(1+2\alpha)(\Gamma(2-\beta))^2} + \frac{2\epsilon^2\ (x+y)(xy)^{1-\beta}}{\Gamma(1+2\alpha)(\Gamma(2-\beta))^2}\ .$$

By the similar manner, the lasting terms can be evaluated.
So by using Eq.(43), Eqs. (45) and Eq. (14), the solution in the series form will be:

$$w(x,y,t) = w_0(x,y,t) + w_1(x,y,t) + w_3(x,y,t) + \cdots$$

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$$w(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \tag{46}$$

LVIM Solution of Application 3

By using the initial conditions (43) within Eqs.(21), the first few iterations of LVIM solution are:

$$w_0(x,y) = x + y,$$

$$w_1(x,y,t) = x + y - \epsilon\left(\frac{x^{1-\beta}}{\Gamma(2-\beta)} + \frac{y^{1-\beta}}{\Gamma(2-\beta)}\right)\ (x+y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{47}$$

$$w_2(x,y,t) = x + y - \frac{\epsilon t^{\alpha} x^{-\beta} y^{-\beta}(x+y)\left(x^{\beta} y + x y^{\beta}\right)}{\Gamma[1+\alpha]\Gamma[2-\beta]},$$

$$+\frac{a^2 t^{2\alpha} x^{-2\beta} y^{-2\beta}(x+y)((x^{2\beta}y^2 + x^2 y^{2\beta} + 4x^{1+\beta}y^{1+\beta})\Gamma[3-2\beta] + (x^{2\beta}y^2 + x^2 y^{2\beta})\Gamma[2-\beta]\Gamma[3-\beta])}{\Gamma[1+2\alpha]\Gamma[3-2\beta](\Gamma[2-\beta])^2}$$

$$-\frac{\epsilon^3 t^{3\alpha} x^{-3\beta} y^{-3\beta}(x+y)\left(x^{\beta} y + x y^{\beta}\right)\Gamma[1+2\alpha]\left(2x^{1+\beta}y^{1+\beta}\Gamma[3-2\beta] + \left(x^{2\beta}y^2 + x^2 y^{2\beta}\right)\Gamma[2-\beta]\Gamma[3-\beta]\right)}{(\Gamma[1+\alpha])^2\Gamma[1+3\alpha]\Gamma[3-2\beta](\Gamma[2-\beta])^3},$$

$$\vdots$$

RDTM Solution of Application 3

By using Eq.(43), then Eqs. (25) take the next form

$W_0(x,y)$By using Eq.(43), then Eqs. (25) take the following form

$$W_0(x,y) = x + y,$$

$$W_1(x,y) = -\frac{\epsilon(x+y)}{\Gamma(\alpha+1)}\left(\frac{x^{1-\beta}}{\Gamma(2-\beta)} + \frac{y^{1-\beta}}{\Gamma(2-\beta)}\right), \tag{48}$$

$$W_2(x,y) = \frac{\epsilon^2\Gamma[3-\beta](x+y)\left(x^{2-2\beta} + y^{2-2\beta}\right)}{\Gamma[1+2\alpha]\Gamma[3-2\beta]\Gamma[2-\beta]} + \frac{\epsilon^2(x+y)\left(x^{1-\beta} + y^{1-\beta}\right)^2}{\Gamma[1+2\alpha](\Gamma[2-\beta])^2}$$

$$+\frac{2\epsilon^2(x+y)(xy)^{1-\beta}}{\Gamma[1+2\alpha](\Gamma[2-\beta])^2},$$

$$\vdots$$

By the similar manner, the lasting terms of the series Eqs. (25) can be determined. The series solution Eq.(26) takes the form

$$w(x,y,t) = \sum_{k=0}^{\infty} W_k(x,y) t^{k\alpha}. \tag{49}$$

Numerical Study of Application 3

Table 3 tabulates the $L2$-norm errors, $\sqrt{\sum_{i=0}^{n}\left(w_{approx}(x_i,y_i,t) - w_{exact}(x_i,y_i,t)\right)^2 h}$, of the three anticipated techniques at $0 \le x,y \le 1, h = 0.1,\ \epsilon = 0.1,\ \mu = 1$ at dissimilar values of t. It is notable that in application 3, the evaluated solutions achieved by LVIM converge quicker than the approximate solution by LADM and RDTM. It is also obvious that the effectiveness and the accuracy of these approaches can be significantly improved by calculating more terms of LADM or LVIM or RDTM. It's

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worthy mentioned that the numerical results in Table 3 are obtained by using three iterations only of the proposed schemes. To illustrate the precision of the anticipated techniques, graphical representations of the analytical approximate solutions of $w(x,y,t)$ for various values of the parameters ϵ ($\epsilon = 0.1$ and 0.01) when $\mu = 1$ $y = 0.1$, $0 \leq x \leq 1$, $0 \leq t \leq 1$. In Figures 9 and 10, the exact solution is shown. Figure 11 compares the behaviour of the computed $w(x,y,t)$ solutions using LADM, LVIM, and RDTM with the precise solution at various α ($\alpha = 1$, 0.95, 0.85 and 0.65) and β ($\beta = 1$, 0.9, 0.7 and 0.5) values. When the value of the fractional order approaches unity, these graphs indicate that the obtained solutions cover the traditional results. Furthermore, the suggested techniques are precise and effective. They also have an excellent understanding of one other's perspectives.

Table 3: The $L2$-Norm Error for The Proposed Methods for Application 3 at: $y \leq 1, 0 \leq x, h = 0.1, \epsilon = 0.1, \mu = 1$.

L2-norm of the suggested methods			
t	LADM	LVIM	RDTM
0.02	3.2E − 10	1.1E − 10	3.2E − 10
0.1	1.9E − 07	6.4 E − 08	1.9E − 07
0.2	3.1E − 06	9.8 E − 07	3.1 E − 06
0.3	1.5E − 05	4.8 E − 06	1.5E − 05
0.4	4.7E − 05	1.4 E − 05	4.7 E − 05
0.5	1.1E − 04	3.4 E − 05	1.1 E − 04
0.6	2.3E − 04	6.8 E − 05	2.3 E − 04
0.7	4.2E − 04	1.2 E − 04	4.2E − 04
0.8	7.0E − 04	2.0 E − 04	7.0 E − 04
0.9	1.1E − 03	3.1 E − 04	1.1 E − 03
1	1.7E − 03	4.5 E − 04	1.7 E − 03

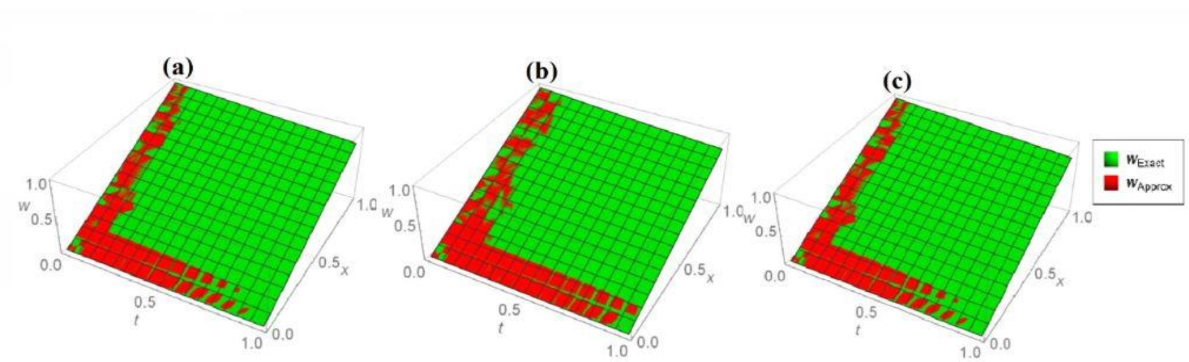

Fig. 9: The space-time surfaces of the approximate solutions w_{approx} of application 3 obtained by LADM, LVIM and RDTM respectively, for $\epsilon = 0.1$, $\mu = 1$, $0 \leq x, t \leq 1$ $y = 0.1$ and the exact solution.

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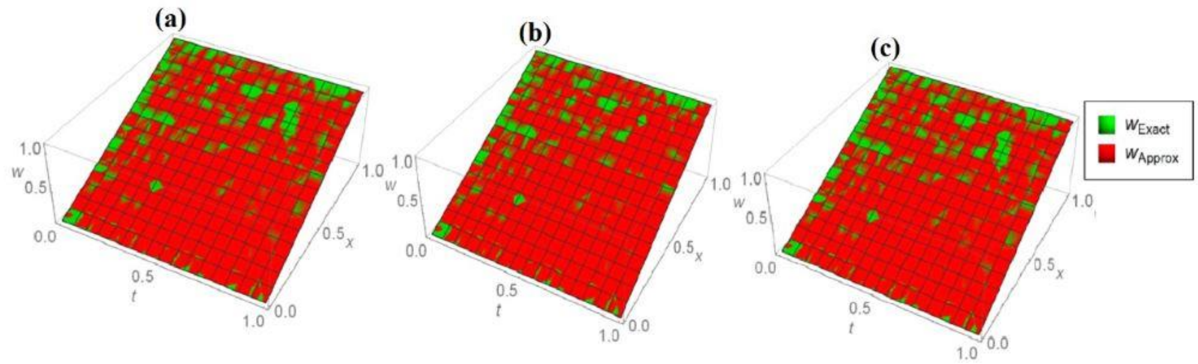

Fig. 10: The space-time surfaces of the approximate solutions w_{approx} of application 3 obtained by LADM, LVIM and RDTM respectively, for$\epsilon = 0.01,\ \mu = 1,\ 0 \leq x, t \leq 1\ \ y = 0.1$ with the exact solution.

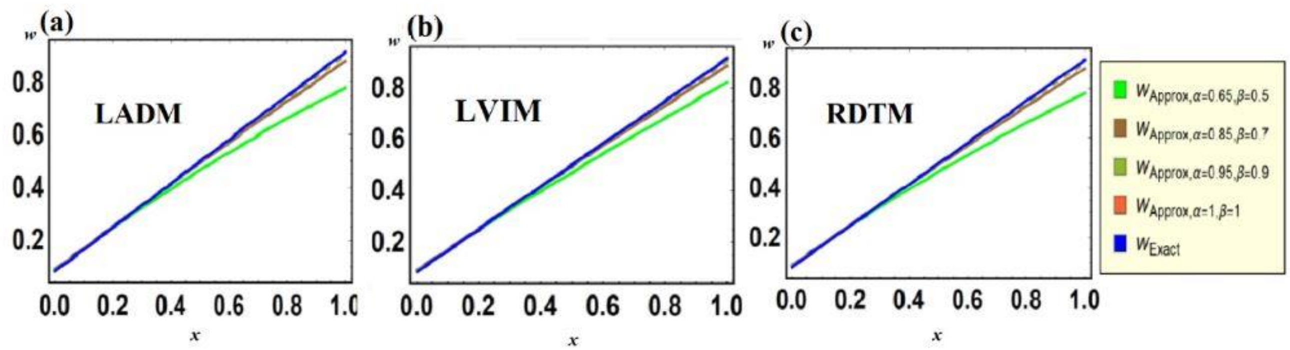

Fig. 11: The x-direction of curves of w_{approx} of application 3 obtained by LADM, LVIM and RDTM respectively, at different values of α and β when $\epsilon = 1,\ y = 0.1,\ \mu = 1, t = 0.1$.

Application 4: Consider the subsequent 2D- time fractional Burger equation (2D-TFBE):

$$D_t^{\alpha} w(x, y, t) + w\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right) = \mu\left(\frac{\partial^2 w}{\partial^2 x} + \frac{\partial^2 w}{\partial^2 y}\right). \tag{50}$$

With the initial condition

$$w(x, y, 0) = (1 + e^{[\frac{x+y}{2\mu}]})^{-1}. \tag{51}$$

The exact solution of Eq. (50) at $\alpha = \beta = 1$ is

$$w(x, y, t) = \left(1 + e^{[\frac{x+y-t}{2\mu}]}\right)^{-1}. \tag{52}$$

LADM Solution for Application 4

By using the initial condition (51), Eqs. (15) take the form

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$$\begin{aligned} f_0(x,y) &= (1+e^{[\frac{x+y}{2\mu}]})^{-1}, \\ f_1(x,y) &= \frac{e^{[\frac{x+y}{2\mu}]}(1+e^{[\frac{x+y}{2\mu}]})^{-2}}{2\mu}, \\ f_2(x,y) &= \frac{e^{[\frac{x+y}{\mu}]}-e^{[\frac{x+y}{2\mu}]}(1+e^{[\frac{x+y}{2\mu}]})^{-3}}{4\mu^2}. \end{aligned} \tag{53}$$

In the same way, the rest of the terms can be determined. Then by using Eqs.(51), (53) and (14), the series solution will be

$$w(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^{\alpha}}{\Gamma(1\alpha+1)} + f_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \tag{54}$$

LVIM Solution of Application 4

By using the initial conditions (51), the first few terms of the solution will

$$\begin{aligned} w_0(x,y) &= \left(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-1}, \\ w_1(x,y,t) &= \left(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-1} + \left[\frac{\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]})^{-2}}{2\mu}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ w_2(x,y,t) &= \left(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-1} + \left[\frac{\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\left(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-2}}{2\mu}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \left[\frac{\left(\mathrm{e}^{\left[\frac{x+y}{\mu}\right]}-\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)\left(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-3}}{4\mu^2}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \left[\frac{\mathrm{e}^{\left[\frac{x+y}{\mu}\right]}(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]})(1+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]})^{-5}(2\alpha+1)}{4\mu^3(\Gamma(\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{55}$$

RDTM Solution of Application 4

Using Eqs. (51), the first few terms of Eqs. (25) take the form

$$\begin{aligned} W_0(x,y) &= \left(1+\mathrm{e}^{[\frac{x+y}{2\mu}]}\right)^{-1}, \\ W_1(x,y) &= \frac{1}{\Gamma(1+\alpha)}\left(0.5\ \mathrm{e}^{[\frac{x+y}{2\mu}]}\left(1.+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}\right)^{-2}\right), \\ W_2(x,y) &= \frac{1}{\Gamma(2\alpha+1)}\left[\frac{\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]}(-0.25+0.25\mathrm{Exp}\left[\frac{x+y}{\mu}\right])(1.+\mathrm{e}^{\left[\frac{x+y}{2\mu}\right]})^{-4}}{\mu^2}\right]. \end{aligned} \tag{56}$$

By similar manner, the excess terms of the series solution Eqs. (25) can be determined. Then Eq.(26) takes the form

$$w(x,y,t) = \sum_{k=0}^{\infty} W_k(x,y)t^{k\alpha}. \tag{57}$$

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Numerical Study of Application 4

Table 4 tabulates the $L2$-norm errors of application 4 at various values of μ and N for the suggested methods at $0 \le x, y \le 1$, and different values of t with the Chebyshev collocation method (CCM) **(Khater et al., 2008)**. The numerical results show that the estimated solutions achieved by our proposed methods converge quicker than the approximate solution by CCM **(Khater et al., 2008)**. It is apparent that the efficacy of these attitudes can be significantly improved by calculating more terms. The t-direction of curves of w_{approx} attained by offered methods when $\mu = 1$, $\alpha = 1$, with exact solution is illustrated in Figure 12. Figure 13 illustrates the behaviors of the computed solutions of $w(x, y, t)$ consuming the proposed approaches at several values of $\alpha, (\alpha = 1, 0.95, 0.85, 0.65$ and $0.45)$ with the exact solution.

Table 4: Comparison of The $\boldsymbol{L2}$-Norm for The Suggested Methods with The CCM **(Khater et al., 2008)** at Various Values of $\boldsymbol{\mu}$ and $\boldsymbol{N}$ for Application 4 at: $\boldsymbol{0 \le x, y \le 1}$

The Suggested Methods					
t	μ	N	LADM	LVIM	RDTM
0.05	1	3	$2.1E-09$	$5.2E-09$	$2.1E-09$
		4	$2.0E-11$	$8.2E-11$	$2.0E-11$
	0.1	3	$2.1E-05$	$5.5E-05$	$2.1E-05$
		4	$2.0E-06$	$9.0E-06$	$2.0E-06$
0.25	1	3	$1.3\ E-06$	$3.3E-06$	$1.3E-06$
		4	$6.4E-08$	$2.7E-07$	$6.4E-08$
	0.1	3	$1.2E-02$	$4.5E-02$	$1.2E-02$
		4	$6.0E-03$	$3.5E-02$	$6.0E-03$
The $L2$-Norm Errors for Various of μ and (N, N) for CCM (Khater et al., 2008)					
0.05	1	(10,10)	$7.5E-07$		
	0.1	(10,10)	$1.3E-06$		
0.25	1	(10,10)	$8.1E-07$		
	0.1	(10,10)	$2.1E-06$		

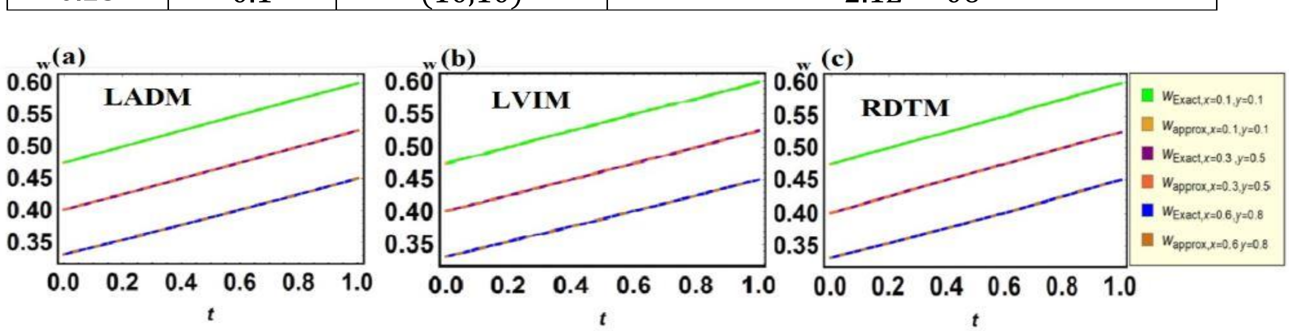

Fig. 12: The t-direction curves of w_{approx} of application 4 gained by LADM, LVIM and RDTM respectively, when $\alpha = 1$, $\mu = 0.1$ and the exact solution.

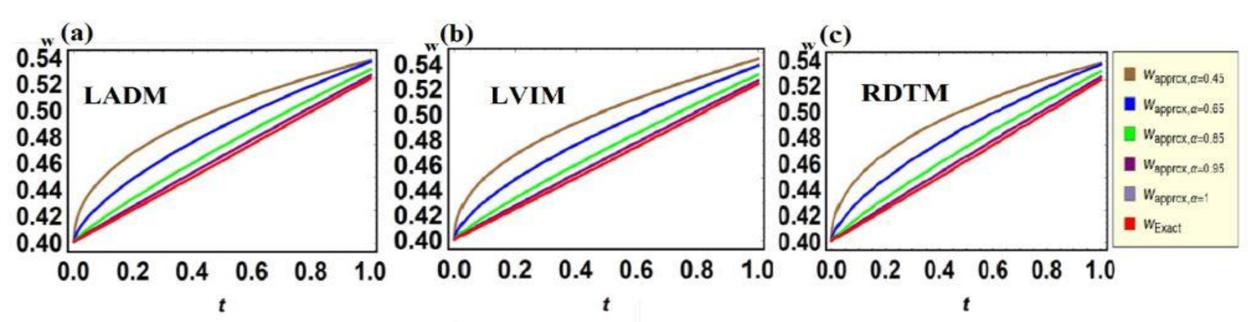

Fig.13: The t-direction of curves of w_{approx} of application 4 achieved by LADM, LVIM and RDTM respectively, at various α values when $y = 0.7,\ \mu = 1, x = 0.1$ and the exact solution.

Conclusions

In this work, three effective semi-analytical methods including the LADM, LVIM, and RDTM have been effectively implemented to solve the (1+n)D-STFBE. The solution was presented as a convergent series with easily calculable components using the approaches. Numerical analyses of four different models of fractional Burgers' equation in one and two-dimensional spaces were used to demonstrate the efficacy of the proposed approaches. The numerical comparisons of the approaches in **(Khater et al., 2008; Doha et al., 2013; Yokus, 2018)** with the suggested procedures and the exact solution are exhibited. It is clear that only a few components of the LADM, RDTM, and LVIM series solutions were adequate for evaluating successful approximation solutions for the 1D and 2D Burgers' equation. It is obvious that the efficacy of the offered approaches can be enhanced by computing more components or terms, and it is noteworthy that the solutions are influenced by time-fractional derivatives. Furthermore, the results obtained by the three methods given were sufficient to obtain accurate solutions to the aforementioned problems and were more accurate than the results in **(Khater et al., 2008; Doha et al.,, 2013; Yokus, 2018).**

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Non

Disclosure statement

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Availability of data and materials

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Contributions

The authors have made each part of this paper. they read and approved the final manuscript.

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References:

Abbaoui, K. and Cherruault, Y., (1995): New ideas for proving convergence of decomposition methods, Comput. Math. Appl., 29, 103-108.

Adomian, G. and Rach R., (1992): Nosi terms in

DJS Vol. 44 (2) (2022) – pp.101-123 ISSN: 1012-5965

decomposition series solution, Comput. Math. Appl., 24(11).

Ahmed, H., (2019): Efficient methods for the analytical solution of the fractional generalized Fisher equation, J .Frac. Cal.and App., 10(1), 85-104.

Ahmed, H., (2020): Analytical approximate solutions for 1D AND 2D nonlinear fractional diffusion equations of type, Comptes rendus de l'Académie bulgare des Sciences 73.3.

Ahmed, H., Bahgat, M.S.M. and Zaki M., (2020): Numerical study of multidimensional fractional time and space coupled Burgers' equations, Pramana , 94(1), 1-22.

Ahmed, H., Bahgat, M.S.M. and Zaki, M., (2017): Numerical approaches to system of fractional partial differential equations, Journal of the Egyptian Mathematical Society, 25(2),141-150.

Ahmed, H., Bahgat, M.S.M. and Zaki, M., (2019):, Analytical approaches to space-and time-fractional coupled Burgers' equations, J. Pramana, 92(3), 1-14.

Das A., (1994): Detailed study of complex flow fields of aero dynamical configurations by using numerical methods, Sadhana , 19(3), 361-399.

Debnath, L., (2003): Fractional differential equations in fluid mechanics, J. Frac. Cal., Appl. Anal.

Doha, E.H., Baleanu, D., Bhrawy, A.H. and Abdelkawy, M.A., (2013): A Jacobi collocation Method for solving nonlinear Burgers-type equations, Abstract and Applied Analysis, Vol. 2013, p.12.

Dutta, P., Saha, S.K. and Nandi, N., (2016): Numerical study on flow separation in 90 pipe bend under high Reynolds number by k – ε modeling, Int. J. Eng. Sci. Technol, 19(2), 904–910.

El-Kalla, I.L., (2007): Errors' Analysis of Adomian series solution to a class of nonlinear differential equations, Appl.Math.E-Notes, 7, 214-221.

Garra, R., (2011): Fractional-calculus model for temperature and pressure waves in fluid-saturated prous rocks. Phys. Rev. E., 84,No3,

Alqhtani, M., Saad, K.M., Shah, R., Weera, W and Hamanah, W.M., (2022): Analysis of the Fractional-Order Local Poisson Equation in Fractal Porous Media, Symmetry, 14, 1323.

Babolian, B. and Biazaar, J., (2002): On the order of convergence of Adomian method, Appl. Math.Comput.,130, 383-387.

Bateman, H., (1915): Some recent researches on the motion of fluids, Monthly Weather Rev., 43,163-170.

Bhrawy, A.H., Zaky, M.A. and Balean, D., (2015): New numerical approximations for space- time fractional Burgers' equations via a Legendre spectral-collocation method, Ramanian Reports in Physics, 67(2), 340-349.

Burgers, J.M., (1948): A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech. 1, 171-199.

Cherruault, Y., Adomian, G., Abbaoui, K. and Rach, R., (1995): Further remarks on convergence of Adomian method, Bio-Med.Comput.38, 89-93.

036605, 1-6.

Guo-Cheng Wu and Dumitru Baleanu, (2013): Variational iteration method for fractional calculus-universal approach by Laplace transform, Advances in Difference Equations. Adv. Differ. Equ. 2013:18.

INAN, B. and BAHADIR, A. R., (2013): Numerical solution of the one dimensional Burgers' equation and fully implicit exponential finite difference methods, Pramana, 547-556.

Inc, M., (2008): The approximate and exact solutions of the space and time fractional Burgers' equations with initial conditions by variational iteration methods, J. Math. Anal. Appl., 345, 475- 484.

Jafari, H., Nazari, M., Baleanu, D., Khalique, C.M., (2013): A new approach for solving a system of fractional partial differential equations, Comput. Math. Appl., 66, 838-843.

Jiang, Y.M., Wei, T.T. and Zhou, X.W., (2012): Stochastic generalized Burgers equations driven by fraction noises. J. Diff. Eq., 252, No2, 1934-1961.

DJS Vol. 44 (2) (2022) – pp.101-123 ISSN: 1012-5965

Khater, A.H., Temsah, R.S. and Hassan, M.M., (2008): A Chebyshev spectral collocation method for solving Burgers-type equations, J. Comput. and Applied Math., 222, 333-350.

Khuri, S.A., (2001): A Laplace decomposition algorithm applied to class of nonlinear differential equations, J. Math. Appl., 4, 141-155.

Kilicman, A., Shokhanda, R. and Goswami, P., (2021): On the solution of (n+1)- dimensional fractional M-Burgers' equation, Alex. Eng. J., 60, 1165-1172.

Lax, P.D., (1973): Hyperbolic systems of conservation laws and the mathematical theory of shock waves, SIAM.

Li, C., Qian, D. and Quan Chen Y., (2011): On Riemann Liouville and Caputo Drivative, Discrete Dynamics in Nature and Society, 1-15.

Miller, K.S. and Ross, B., (1993): An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, Inc. Canada.

Momani, S., (2006): Non-perturbative analytical solutions of the space- and time-fractional Burgers' equations, Chaos Solitons Fractals 28(4), 930–937.

Moslem, W.M., Sabry, R. and Zakharov, K., (2008): Burgers' equation for dust ion acoustic waves, Chaos Solitons Fractals, 36, 628-634.

Obibat Zaid M., (2010): A study on the convergence of variational iteration method, Math. Comput. Modelling, 51, 1181-1192.

Odibat, Z. M., Kumar, S., Shawagfeh, N., Alsaedi, A. and Hayat , T., (2016): A study on the convergence conditions of generalized differential transform method, Mathematical Method in the Applied Sciences.

Rashidi, M.M. and Erfani, E., (2009): New analytical method for solving Burgers' and nonlinear heat transfer equations and comparison with HAM, Comput. Phys. Commun, 180, 1539-1544.

Ray, S.S, (2014): New approach for general convergence of the Adomian decomposition method, World Applied Sciences Journal, 32(11), 2264-2268.

Saad, K.M. and Al-Sharif, E.H., (2017): Analytical study for time and time-space fractional Burgers' equation. Adv. Differ. Equ. 2017, 300.

Sabatier, J., Agrawal, O.P. and Machado, J., (2007): Theoretical developments and applications in physics and engineering, Advances in Fractional Calculus, Springer, Dordrecht.

Sohail, M. and Mohyud-Din, S.T., (2012): Reduced differential transform method for solving a system of fractional PDEs, International Journal of Modern Mathematical Sciences, 4(1), 21-29.

Sripacharasakullert, P., Sawangtong, W. and Sawangtong, P., (2019): An approximate analytical solution of the fractional multi-dimensional Burgers' equation by the homotopy perturbation method, Advances in Difference Equations, 2019:252.

Srivastava, H.M., Saad , K.M. and Hamanah , W.M., (2022): Certain New Models of the Multi-Space Fractal-Fractional Kuramoto-Sivashinsky and Korteweg-de Vries Equations, Mathematics, 10, 1089.

Wazwaz, A.M., (2010): The combined Laplace transform Adomain Decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Math.Compu.,216,1304 1309.

Yokus, A. and Kaya, D., (2017): Numerical and exact solution for time Burgers' equation, J. Nonlinear Sci. Appl. ,10, 3419-3428.

Yokus, A., (2018): Numerical solution for space and time fractional order Burger type equation, Alex. Eng. Journal 57, 2085-2091.

Zedan, H.A., Tantawy, S.SH., Sayed, Y.M., (2014): Convergence of variational iteration method for initial-boundary value problem of fractional integro-differential equations, J. Frac. Cal. and App., Vol.5 (3s), No. 26, 1-14.

Zhhou, J.K., (1986): Differential transform and its applications for electrical circuits, Huazhon University Press, Wuhan, China.

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الملخص العربي

دراسة مقارنة لطرق فعالة لحل نموذج جديد (ن + 1) من الأبعاد لمعادلة برجر ذات الرتبة الكسرية

محمد سباق محمد بهجات، هدى فرغل احمد و مفيدة على زكى

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يقدم العمل الحالي حلول تقريبية لنموذجًا جديدًا (ن + 1) من الأبعاد لمعادلة برجر ذات الرتبة الكسرية ويقدم دراسة تحليلية مقارنة لثلاث طرق شبه تحليلية فعالة لحل معادلة برجر من الرتبة الكسرية. الطرق المستخدمة هي طريقة لابلاس- أدوميان التحليلية Laplace Adomian decomposition method (LADM) ، طريقة لابلاس- التغير التكراري Laplace variational iteration method(LVIM)و طريقة إختزال التحولية التفاضلية Reduce differential transform method (RDTM). الطرق المقترحة تسخدم الشروط الأولية فقط لايجاد الحلول دون أي تقدير أو تقييد. علاوة على ذلك ، تكون هذه الحلول في شكل متسلسلة لانهائيه تقاربيه ، كل حد فيها يمكن حسابه أو إيجاده بسهول. يتم تقديم دراسات تحليله وعددية لأربعة تطبيقات من معادلة برجر من الرتبه الكسرية لتوضيح فاعليه ودقة الطرق المستخدمه، وكذلك لمقارنة الطرق مع بعضها البعض ومع الطرق الأخرى المستخدمة في الدراسات المتاحة الأخرى. بالإضافة إلى توضيح خصائص الحلول للتطبيقات الاربعة عند تغيير رتبة التفاضل الكسرى. كما تظهر النتائج العددية فعالية ودقة الطرق المستخدمة.