ON A CERTAIN UNIFORM STRUCTURE
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ABSTRACT
A regular structure for a topological space is defined. Also, a relation between a regular structure and an Abian's structure is investigated.

INTRODUCTION
In [1], Abian defined a uniform structure for a topological space so that the theorem that a continuous function on a compact space is uniformly continuous holds. As uniform spaces are well known [e.g. 1, 4], one would naturally like to compare a uniform structure with a uniformity when the former is defined. However, it does not seem easy to do this between a uniform structure defined by Abian (we shall use the name “Abian’s structure” in the sequel) and a uniformity. In this paper, we define a regular structure (Definition 2.1 below) for a space so that the above mentioned theorem holds and it is easy to compare a regular structure with a uniformity. Moreover, a relation between a regular structure and an Abian’s structure is found and it follows that a topological space which admits an Abian’s structure must be regular.

1. PRELIMINARIES
We state here definitions occurred in [1] and a simple result for later use. Let $X$ be a topological space and $I$ an index set. For each $i \in I$, let $\mathcal{G}_i$ be a collection of open sets of $X$. Moreover, for each $i \in I$ and each $x \in X$, let $\chi_i(x) = \{X \in \mathcal{G}_i : x \in X\}$ and $\Sigma_i(x) = \chi_i(x) : X \in \mathcal{G}_i(x)$. An ordering $\leq$ for $I$ is defined as follows. For $i, j \in I$, $i \leq j$ if and only if $\forall x \in X \in \mathcal{G}_j$. It should be noted that $\leq$ is reflexive and transitive. The family $\{\chi_i : i \in I\}$ is called an Abian’s structure for the space $X$ if it has the following properties (G), (B) and (M).

(i) If $x$ is a point of an open set $G$ in $X$, then there is an $i \in I$ such that $\chi_i(x) \subseteq G$.
(ii) For any $i$ in $I$, we have either $i \leq j$ or $j \leq i$.
(iii) $\chi_i(x) \cap \chi_j(x) \neq \emptyset$ for each $i \in I$.

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Next property (V) is an easy consequence of (I) and the fact that each member
in \( \mathcal{U} \) is open in \( X \).

(V) A subset \( G \) of \( X \) is open if and only if for each \( x \in G \) there is an \( I \in \mathcal{I} \)
such that \( S(x) \subseteq G \). Now, let \( X \) and \( Y \) be two topological spaces with Abian's structures \( (\mathcal{U}, \mathcal{V}) \) and \( (\mathcal{W}, \mathcal{V}) \) respectively. A function \( f \) from \( X \) to \( Y \) is said to be uniformly
continuous if for each \( \epsilon > 0 \), \( \mathcal{V} \) is mapped into some \( \mathcal{V} \) in \( \mathcal{V} \).

2. DEFINITIONS

In this section we define regular structures for a non-empty set and introduce
the concept of uniformly continuous functions.

Definition 2.1. Let \( X = (X, \mathcal{U}) \) be given and \( \Delta \) denote the diagonal of \(! \times \! X \). A
collection \( \mathcal{U} \) of subsets of \( X \times X \) is called a regular structure for \( X \) if the following
conditions are fulfilled:

1. \( \Delta \) is a subset of each \( U \in \mathcal{U} \).
2. The transpose \( U^T \) of each member \( U \in \mathcal{U} \) contains some \( V \in \mathcal{U} \).
3. The intersection \( U \cap V \) of each two members in \( \mathcal{U} \) contains a third one.
4. For each \( U \in \mathcal{U} \) and \( x \in X \), there exists a \( V \in \mathcal{V} \) such that \( V \; V(x) \subseteq U \) for all \( y \in V(x) \).

The above notions are as in [2, 3, 4].

It is clear that \( \mathcal{U} \) would be a base for a uniformity for \( X \) if condition (4) was
suitably strengthened. It is also clear that a topology for \( X \) can be induced by
defining a set \( G \subseteq X \) to be open if for each \( x \in G \) there is a \( U \in \mathcal{U} \) with \( U(x) \subseteq \).

Definition 2.2. Let \( \mathcal{V} \); \( (X, \mathcal{U}) \) and \( (Y, \mathcal{V}) \) be two spaces (with this notation, it
is understood that \( \mathcal{U} \) and \( \mathcal{V} \) are regular structure for \( X \) and \( Y \) respectively) and let \( f \) be a function from \( X \) to \( Y \). \( f \) is said to be uniformly continuous if for each \( V \in \mathcal{V} \),
there exists a \( U \in \mathcal{U} \) such that \( f(U) \subseteq V \) for all \( x \in X \).

3. RESULTS

Theorem 3.1. If a function \( f \) from \( \mathcal{U} \times \mathcal{U} \) to \( \mathcal{V} \times \mathcal{V} \) is continuous and the
space \( \mathcal{U} \times \mathcal{V} \) is compact (topological properties are relative to the topologies
induced by \( \mathcal{U} \) and \( \mathcal{V} \)), then \( f \) is uniformly continuous.

Proof. Let \( V \in \mathcal{V} \) be given. For each \( x \in X \), since \( f(x) \in \mathcal{V} \), by condition (4)
in 2.1, there exists a \( V \in \mathcal{V} \) such that \( V \subseteq \mathcal{V}(x) \) for all \( y \in V \). Also,
there exists \( y \in \mathcal{V} \) such that \( f(y) \subseteq \mathcal{V}(x) \). Since \( f \) is continuous, there is for
each \( x \in X \); \( U \in \mathcal{U} \) such that \( f(U) \subseteq W \). By conditions (4) in 2.1.
again, there is for each \( x \in X \) a \( U_i \in \mathcal{U} \) such that \( \bigcup_{i} U_i \supseteq \{ x \} \) for all \( x \in X \). Clearly, \( (U_i, \{ x \}) : x \in X \) is an open covering for the compact space \( X \). Thus \( \bigcup_{i} \{ U_i \} = X \) for some finite set \( \{ U_{i_1}, \ldots, U_{i_n} \} \) in \( X \). By condition (3) in 2.1, there exists a \( U \in \mathcal{U} \) contained in \( \bigcap_{i} \{ U_{i_1}, \ldots, U_{i_n} \} \) in \( X \). Moreover, there exists a \( U \in \mathcal{U} \) for all \( x \in X \). To see this, let \( x \in X \) be given. \( x \in \bigcup_{i} U_i \) for some \( p \in \{ 1, 2, \ldots, n \} \). Hence \( U \in \mathcal{U} \). This together with \( (\ast) \) gives \( U \subseteq \bigcup_{i} U_i \) for all \( x \in X \). This proof is complete.

To study the relation between an Abian's structure and a regular structure, let \( X \) and an Abian's structure \( (x_i : i \in I) \) be given. For each \( i \in I \), we define \( U_i = (x_{i+}, y \in S(x)) \). With this notation, we have

**Theorem 3.2.** \( \mathcal{U} = (U_i : i \in I) \) is regular structure for \( X \) (we shall call it the regular structure induced by the given Abian's structure) and the topology induced by \( \mathcal{U} \) coincides with the given topology for \( X \).

**Proof:** We have to check if \( \mathcal{U} \) satisfies the four conditions in 2.1. conditions (1) is clearly satisfied. We shall show that \( U_{i_j} \in \mathcal{U} \) for each \( i \in I \) and condition (2) follows. It is easily seen that \( U_{i,i} = S(x) \) for all \( i \in I \) and for all \( x \in X \). If \( U_{i,i} \in \mathcal{U} \), then \( y \neq S(x) \) for any \( y \in S(x) \) (i.e., there is an \( X \subseteq I \) such that \( y \in X \)). Then \( x \in X \) (or \( y \neq S(x) \) which implies \( y \in S(x) \) or \( x \in U_{i,i} \)). To prove that condition (3) holds, let \( i, j \in I \) be given. In view of property (II) of \( \{ S(i) \} \), we may assume that \( i \neq j \) if it is. Then it is clear that \( U_i \cap U_j = U_{i,j} \). Finally, let \( i \neq j \) and \( x \in X \) be given. Fix an \( x \in X \). Since \( X \) is open and \( x \in X \), (by (1)), there is a \( f \) such that \( \gamma (Y) = S(x) \). Similarly, there is a \( f \) (we may assume \( k_y \)) such that \( \gamma (Y) \neq S(x) \) or \( S(x) \). It is routine to show that \( U_{i,j} = \{ x \} \) for all \( x \in U_i \). This condition (4) is also satisfied. It remains to show that the topology induced by \( \mathcal{U} \) is the same as the given one for \( X \). This is obvious by (II) and the fact that \( S(x) = U_{i,i} \) for all \( x \in X \).

Concerning the definition of uniformly continuous function, we have the following:

**Theorem 3.3.** Let \( X, Y \) be topological spaces with Abian's structures \( (x_i : i \in I) \) and \( \{ Y_j : j \in J \} \) respectively, and let \( f \) be a function \( X \) from to \( Y \). We have:

\[ f(X) = \bigcup_{i \in I} \{ f(x_i) \} \]
(a) if f is uniformly continuous according to § 1, then f is uniformly continuous according to Definition 2.2 relative to the regular structures induced by the given Abian's structures.

(b) if the space X is compact, then the converse of (a) is true.

Proof. Let the induced regular structures be denoted by \( \bar{U} = (U_{ij}; i, j) \) and \( \bar{V} = (V_{ij}; i, j) \) respectively where:

\[ U_{ij} = \{ (x, y) : x \in S_i(x) \} \quad \text{and} \quad V_{ij} = \{ (x, y) : y \in S_i(y) \} \]

To prove (a), let \( (x, y) \in \bar{U} \) be given. By assumption there is an \( i \) such that, for each \( x \in S_i(x) \), \( f(x) \in \bar{V} \), for some \( y \in S_i(y) \). Obviously, for any fixed \( x \in S_i(x) \), \( f(x) \in \bar{V} \) for every \( y \in S_i(y) \). Thus, \( f(U_{ij}) = f(S_i(x)) \subseteq \bar{V} \), \( S_i(y) \subseteq \bar{U} \), \( i \) = \( V_{ij} \), \( f(U_{ij}) \subseteq \bar{V} \).

We proceed to prove (b). For given \( i \in J \) and \( x_0 \in X \), fix a \( Y_i \in \bar{V}_i \) (if \( \bar{V}_i \)). Since \( f(x) \in Y_i \) and \( Y_i \) is open in \( X \), there is an \( i \in J \) such that \( V_{ij} \ni f(x_0) \subseteq Y_i \). Uniform continuity of \( f \) according to Definition 2.2 implies the existence of an \( m \) such that \( f(U_{ij}) \subseteq V_{ij} \) for all \( x \in X \). Now, \( x_0 \in X \) and \( U_{ij} \subseteq \bar{U} \), imply that there is a \( x_0 \in X \) such that \( U_{ij} \ni x_0 \subseteq U_{ij} \ni x \). We see that for every \( x \in U_{ij} \ni x_0 \), \( f(U_{ij} \ni x) \subseteq f(U_{ij} \ni x_0) \subseteq \bar{V} \ni f(x_0) \subseteq \bar{V} \). Noting that \( x_0 \) is an arbitrary point in \( X \), we have for each \( x \in X \) a \( (x, y) \in \bar{U} \) such that \( f(U_{ij} \ni x) \subseteq \bar{V} \ni f(x_0) \subseteq \bar{V} \).

Assort that each \( x \in X \) is mapped into some \( Y_i \in \bar{V}_i \). Since \( x \in x_i \) must be a member of \( \bar{V}_i \) for some \( x \in X \) (the case \( x = 0 \) is trivial and is not considered), \( x \in \bar{V}_i \ni S_i(y) \ni x = \bar{V}_i \ni x \), where \( x \) is \( \in \bar{V}_i \ni \{ x \} \), with \( x \in \bar{V}_i \ni \{ x \} \). It follows that \( f(x_0) \in f(U_{ij} \ni x_0) \subseteq \bar{V} \), \( i \) = \( V_{ij} \), \( f(x_0) \subseteq \bar{V} \) for some \( Y_i \in \bar{V}_i \), the theorem is proved.

4. REMARKS

Remark 4.1. Owing to Theorem 2.3, Abian's result follows from our Theorem 3.1. If a regular structure \( \bar{U} \) for a topological space \( X \) (that is, \( \bar{U} \) induces the topology for \( X \) is induced by an Abian's structure), then every two members of \( \bar{U} \) are comparable with respect to \( \bar{U} \). However, in general a regular structure need not have this property and hence is not necessarily induce by an Abian's structure. Thus we can hardly say that Abian's result implies ours.

Remark 4.2. A regular structure for a space \( X \) is similar to a "symmetric indexed neighborhood system with 'local triangle inequality.'

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for $X^*$ defined by Davis [2]. Similar to his proof [2, Theorem 4], we can show that a topological space admits a regular structure if and only if it is regular. Consequently, a topological space that has an Abian's structure must be regular. It is unknown to the author whether a regular topological space admits an Abian's structure or not.

REFERENCES


عند شبكة مؤخّرة منقولة

سواح خلال السيد

قسم الرياضيات – كلية العلوم – جامعة الملك عبد العزيز

في هذا البحث يتم تعريف شبكة مؤقّرة تطبيقي ومن ثمّ دراسة العلاقة بين هذه الشبكة المنقولة وبين شبكة مؤكّرة منقولة.