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ISOMETRIC FOLDING OF FINITE TYPE

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ABSTRACT

In this paper we introduced a new type of isometric folding we called it isometric folding of finite type. We proved that any isometric folding is of finite type iff it determines a stratification on M made up of finitely many strata of dimensions 0 and 1. Finally we explored the connectedness of the singular set and we proved that the composition and the Cartesian product of isometric foldings of finite type is an isometric folding of finite type.

1. INTRODUCTION

The theory of isometric folding is introduced by S. A. Robertson [12], then it has been pushed by E. El-Kholy [2]. Many other types of foldings are invented by E. El-Kholy and others [3:12].

Let M and N be smooth connected Riemannian manifolds, of dimensions m and n respectively such that $m \leq n$. A map $f : M \longrightarrow N$ is said to be an isometric folding of M into N iff, for every piecewise geodesic path $\gamma: J \longrightarrow M$, the induced path $f \circ \gamma : J \longrightarrow N$ is piecewise geodesic and of the same length as γ, [13]. The set of points of M where f fails to be differentiable is called the set of singularities of the isometric folding f and is denoted by $\Sigma(f)$.This set of singularities will partition the manifold M into strata of dimension $k = 0,1,\ldots, m$, where $\Sigma_k(f)$ denote the union of all strata of dimension k. This set corresponds to the folds of the map. The set of all isometric foldings of M into N is denoted by $\Im(M, N)$ and of M into itself by $\Im(M)$.

There is no assumption about continuity or differentiability of isometric folding, continuity follows from the definition. In general an isometric folding need not be differentiable.

Let $f \in \Im (M, N)$, where M and N are smooth Riemannian 2-manifolds, i.e., surfaces, then the set of singularities, $\Sigma (f)$, is a graph embedded in M, such that every vertex of the graph has even valency (the valency of a vertex of the graph is the number of edges incident at the vertex in the graph). Of course, $\Sigma (f)$ need not be connected and may have components homeomorphic to circles and have no vertices [12].

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2. RESULTS

(2.1) Definition

An isometric folding $f \in \Im(M, N)$ is said to be of **finite type** iff f has finitely many regions.

We denote the set of all isometric foldings from M into N of finite type by $F(M, N)$, and the set of all isometric foldings from M into itself by $F(M)$.

A very simple example of an isometric folding of finite type is the map $f : \Re^2 \longrightarrow \Re^2$ given by $f(x, y) = (|x|, |y|)$, see Fig. (1).

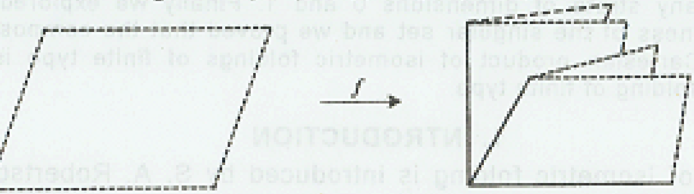


Fig. (1)

From now on, M and N are complete and connected 2-Riemannian manifolds.

(2.2) Theorem

Let $f \in \Im(M, N)$. If F is a face of M, then for all $x, y \in F$ there is a geodesic segment σ on F joining x to y such that $d_F(x, y) = L(\sigma)$. Moreover, if ρ is any geodesic segment on M joining $x \in F$ to $y \in F$ and $L(\rho) = d_M(x, y)$ then ρ is a geodesic segment on F and $d_F(x, y) = d_M(x, y)$.

Proof

Let $f \in \Im(M, N)$ and F be a face of M determined by f. For each $x, y \in F$ there is a geodesic $\tau : \Re \longrightarrow M$ on M such that $\tau(0) = x$, $\tau(t_0) = y$ for some $t_0 \in \Re^+$ and $L(\tau|[0, t_0]) = d_M(x, y)$.

Now, $\tau(t) \in F$ for all $t \in [0, t_0]$, otherwise the induced zig-zag $f \circ \tau : [0, t] \longrightarrow N$ must have singularities on $]0, t_0[$, and hence

$$d_N(f(x), f(y)) < d_M(x, y) \qquad (3)$$

On the other hand $f|F$ is an isometric immersion into N. Thus $d_F(x, y) = d_N(f(x), f(y))$ for all $x, y \in F$. But $d_M(x, y) \leq d_F(x, y)$

Hence $d_M(x, y) \leq d_N(f(x), f(y))$. (4)

which contradics (3). Thus $\sigma = \tau|[0, t_0]$ is a minimising geodesic segment on F, i.e., σ is a geodesic segment and $L(\sigma) = d_F(x, y)$.

By the same procedure we can prove the second part of this theorem.

(2.3) Theorem

Let $f \in \Im(M, N)$. Then $f \in F(M, N)$ iff it determines a stratification on M made up of finitely many q-strata, $q = 0,1$.

Proof

Let f be an isometric folding with a stratification on M made up of finitely many q-strata, $q = 0,1$. Since any i-strata, $i = 0,1$, of M must belong to the frontier of at least two distinct $(i+1)$ strata. Thus no. $(\Sigma_2 f) \leq 2$ (no. $\Sigma_1 f$). Then the number of strata of dimension two is finite and hence the isometric folding is of finite type.

For the converse, let f be an isometric folding of finite type. Then the number of strata of dimension two is finite, say r. Let $A_1, A_2, \ldots, A_r$ be the faces of M.

For each $j = 1,2, \ldots, r$ the frontier, ∂A_j, of A_j is a union of strata of lower dimensions, i.e., $\partial A_j = \cup (C_\alpha)$, $\alpha \in \Omega_j$, where $C_\alpha \in \Sigma_1 f \cup \Sigma_0 f$ and Ω_j is a set of indices. Then either

(i) Ω_j is a finite set for all $j \in \{1,2, \ldots, r\}$ or

(ii) Ω_j is not finite for some $j \in \{1,2, \ldots, r\}$.

In the first case M will be composed of finitely many q-strata, $q = 0,1$.

The second case can not be arised by using the fact that any i-strata, $i = 0,1$, of M must belong to the frontier of at least two distinct $(i+1)$ strata.

In the following two theorems we will explore the connectedness of the singular set of an isometric folding.

(2.4) Theorem

If $f \in \Im(M, N)$ is such that $\Sigma(f) \neq \varnothing$, then $\Sigma(f)$ is connected iff $\Sigma(f)$ is an Eulerian graph.

Proof

As mentioned before, the valency of each vertex of $\Sigma(f)$ for any $f \in \Im(M, N)$ is even. The proof of the theorem follows from the fact that any connected graph is Eulerian iff the degree of each vertex is of even valency [1].

It should be noted that this theorem is also true for isometric foldings of finite type.

(2.5) Theorem

The composite of finite number of isometric foldings of Riemannian manifolds of finite type is an isometric folding of finite type.

Proof

Let M, N and L be any complete connected Riemannian 2-manifolds. Let $f : M \longrightarrow N$ and $g : N \longrightarrow L$ be isometric foldings of finite type, then each of f and g must has finite number of faces, say $A_1, A_2, \ldots, A_s$ and $B_1, B_2, \ldots, B_r$ of M and N respectively.

Now, $A_k \cap A_l = \emptyset$ for $k \neq l$ and $f \mid A_i$ is an isometric immersion of A_i into N. Thus $x \in \Sigma_2\ g \circ f$ iff $x \in A_i \cap f^{-1}\{B_j \cap f(A_i)\}$ for some $i \in \{1, \ldots, s\}$ and some $j \in \{1, \ldots, r\}$.

The subsets $B_j \cap f(A_i)$ of N are arcwise connected sets and their number is finite, hence the number of 2-strata on M determined by $g \circ f$ is less than or equal to rs, and hence finite, i.e., $g \circ f \in F(M, L)$.

(2.6) Example

Let $M = S^2 = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$. Define a map $f : S^2 \longrightarrow S^2$ by $f(x, y, z) = (x, y, |z|)$. Then f is an isometric folding of finite type, the image $f(S^2) = N$ is the upper hemisphere, and the number of regions of f, 2-strata, is two.

Now, let $g : N \longrightarrow L$ be a map defined by, $g(x, y, z) = (|x|, |y|, z)$. Then the map g is an isometric folding of finite type, the image $g(N) = L$ is the positive octant and the number of regions of g is four. Now, $g \circ f : M \longrightarrow L$ is an isometric folding of finite type defined by $(g \circ f)(x, y, z) = (|x|, |y|, |z|)$, and the number of regions of $g \circ f$ is eight, see Fig. (2).

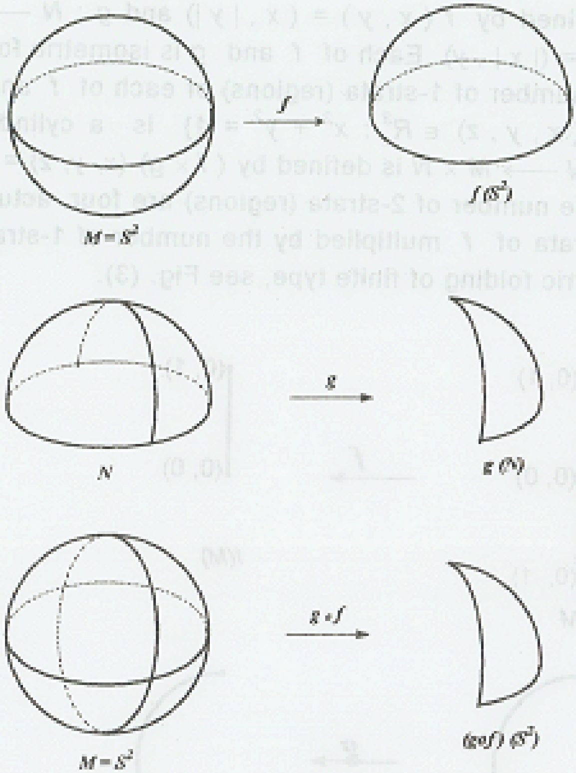


Fig. (2)

(2.7) Theorem

Let $f \in F(M, N)$ and $g \in F(L, P)$. Then $f \times g \in F(M \times L, N \times P)$, i.e., the Cartesian product of isometric foldings of finite type is an isometric folding of finite type.

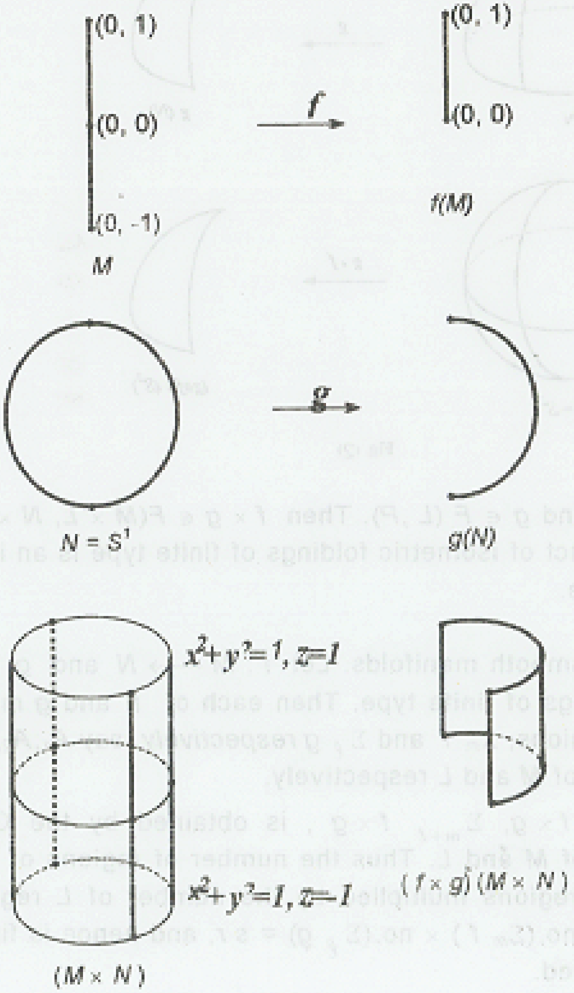
Proof

Let *M, N, L, P* be smooth manifolds. Let $f : M \longrightarrow N$ and $g : L \longrightarrow P$ be isometric foldings of finite type. Then each of f and g must have finite number of regions, $\Sigma_m\, f$ and $\Sigma_\ell\, g$ *respectively, say* $A_1, A_2, \ldots, A_s$ and $B_1, B_2, \ldots, B_r$ of *M* and *L* respectively.

Any region of $f \times g$, $\Sigma_{m+\ell}\ f \times g$, is obtained by the Cartesian product of region of *M* and *L*. Thus the number of regions of $M \times L$ is the number of *M* regions multiplied by the number of *L* region, i.e., on.$(\Sigma_{m+\ell}\ f \times g)$ = no.$(\Sigma_m\ f)$ × no.$(\Sigma_\ell\ g) = s\, r$, and hence is finite, and the theorem is proved.

(2.8) Example

Let $M = \{(0, y) \in R^2 : -1 \leq y \leq 1\}$, $N = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$. Let $f: M \longrightarrow M$ be defined by $f(x, y) = (x, |y|)$ and $g: N \longrightarrow N$ be defined by $g(x, y) = (|x|, y)$. Each of f and g is isometric folding of finite type and the number of 1-strata (regions) of each of f and g are two. Now $M \times N = \{(x, y, z) \in R^3 : x^2 + y^2 = 1\}$ is a cylinder and $(f \times g)(M \times N): M \times N \longrightarrow M \times N$ is defined by $(f \times g)(x, y, z) = (x, |y|, |z|)$. In this case the number of 2-strata (regions) are four, actually it is the number of 1-strata of f multiplied by the number of 1-strata of g. Thus $f \times g$ is isometric folding of finite type, see Fig. (3).



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(الطي المتقايس من النوع المحدود)

إنتصار الخولى و مها أبو شنب

فى هذا البحث تم تقديم نوع جديد من الطى المتقايس أسميناه بالطى المتقايس من النوع المحدود ثم أثبتنا أنه لأى طى متقايس لا توجد طبقات صفرية إذا لم تتواجد طبقات أحادية وأننا نحصل على الطى المتقايس من النوع المحدود إذا وإذا فقط كان لدينا عدد محدود من الطبقات الصفرية والأحادية. وضحنا أيضاً كيفية ترابط مجموعة النقاط الشاذة ثم أثبتنا أن تحصيل وحاصل الضرب الكرتيزى لطيين متقايسين من النوع المحدود هو طى متقايس من النوع المحدود.