Pre - $\theta$ - perfect mappings and $p$ - closed spaces

Abdulla Salem Bin Shanna

Department of Mathematics, University of Aden, Aden, Yemen

(Received: 29-12-2011)

Abstract. In this paper, we introduce Pre - $\theta$ - perfect mappings and investigate some of their characterizations and properties. Also we give a characterization of $p$ - closed spaces.

Key words: Pre - $\theta$ - closed sets, filter base, $p$ - closed spaces.

1. Introduction

A mapping $f: X \rightarrow Y$ is called perfect if $f$ is closed and $f^{-1}(y)$ is compact, for each $y \in Y$. Whyburn [9] proved that a mapping $f: X \rightarrow Y$ is perfect if and only if for every filter base $\Phi$ on $f(X)$ converging to $y \in Y$, $f^{-1}(\Phi)$ is directed towards $f^{-1}(y)$. The purpose of the present paper is to introduce pre - $\theta$ - perfect mappings defined in a way similar to the above characterization of a perfect mapping and investigate some of their properties and characterizations. Also we give a characterization of $p$ - closed spaces.

2. Preliminaries

Recall that a subset $A$ of a space $X$ is called preopen [4] if $A \subseteq \text{int} (\overline{A})$. The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing $A$ is called the preclosure of $A$ and denoted by $\overline{\text{pc}}(A)$.

Definition 2.1[5]. Let $A$ be a subset of a space $X$.

(i) A point $x \in X$ is called a pre - $\theta$ - cluster point of $A$ if $\overline{\text{pc}}(V) \cap A \neq \emptyset$, for every preopen set $V$ containing $x$.

(ii) The set of all pre - $\theta$ - cluster points of $A$ is called the pre - $\theta$ - closure of $A$ and is denoted by $\overline{\text{pc}}_\theta(A)$.

(iii) A subset $A$ of a space $X$ is called pre - $\theta$ - closed if $\overline{\text{pc}}_\theta(A) = A$.

(iv) The complement of a pre - $\theta$ - closed set is called pre - $\theta$ - open.

Remark 2.1. It is obvious that a pre - $\theta$ - open (resp. pre - $\theta$ - closed) set is preopen (resp. preclosed), but the converse need not be true as shown by Example 3.3 of [2].

Definition 2.2. Let $X$ be a topological space.
(1) A point \( x \in X \) is called a pre-\( \theta \) - cluster point of a filter base \( \Phi \) in \( X \) if \( x \in \bigcap \{ \text{pol}_\theta(F) : F \in \Phi \} \).

(2) A filter base \( \Phi \) in \( X \) is \( p\theta \)-* convergent* [2] to a point \( x \in X \) if for each preopen set \( A \) containing \( x \), there exists an \( F \in \Phi \) such that \( F \subseteq \text{pol}(A) \).

(3) A filter base \( \Gamma \) is said to be subordinate [6] to a filter base \( \Phi \) if for each \( F \in \Phi \), there exists \( G \subseteq \Gamma \) such that \( G \subseteq F \).

(4) A filter base \( \Phi \) is said to be \( p\theta \) - directed towards \( A \subseteq X \) if every filter base subordinate to \( \Phi \) has a pre-\( \theta \) - cluster point in \( A \).

3. Pre-\( \theta \) - perfect mappings.

Definition 3.1. A mapping \( f : X \rightarrow Y \) is called pre-\( \theta \) - perfect \( ( p\theta \) - perfect in short) if for every filter base \( \Phi \) in \( f(X) \) \( p\theta \) - convergent to \( y \in Y \), \( f^{-1}(\Phi) \) is \( p\theta \) - directed towards \( f^{-1}(y) \).

Remark 3.1. Continuity is not assumed on \( p\theta \) - perfect mappings.

Definition 3.2. A mapping \( f : X \rightarrow Y \) is called pre-\( \theta \) - closed \( ( p\theta \) - closed in short) if \( \text{pol}_\theta(f(A)) \subseteq f(\text{pol}_\theta(A)) \), for every subset \( A \) of \( X \).

Theorem 3.1. Every \( p\theta \) - perfect mapping is \( p\theta \) - closed.

Proof. Suppose that \( f : X \rightarrow Y \) is \( p\theta \) - perfect mapping. Let \( A \) be any subset of \( X \) and \( y \in \text{pol}_\theta(f(A)) \). Then there exists a filter base \( \Phi \) on \( f(A) \), \( p\theta \) - converging to \( y \). Put \( \Gamma = f^{-1}(\Phi) \cap A : F \in \Phi \). Then \( \Gamma \) is a filter base in \( X \) and subordinate to the filter base \( f^{-1}(\Phi) \). Since \( f^{-1}(\Phi) \) is \( p\theta \) - directed towards \( f^{-1}(y) \), we have \( f^{-1}(y) \cap \{ \text{ad}_\theta(\Gamma) \} = \emptyset \). Therefore, we obtain \( y \in f(\text{pol}_\theta(A)) \). This implies that \( f \) is \( p\theta \) - closed.

Theorem 3.2. A mapping \( f : X \rightarrow Y \) is \( p\theta \) - closed if and only if the image \( f(A) \) of each pre-\( \theta \) - closed subset \( A \) of \( X \) is pre-\( \theta \) - closed subset of \( Y \).

Proof. Obvious.

Theorem 3.3. The composition \( g \circ f : X \rightarrow Z \) of \( p\theta \) - closed mappings \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) is \( p\theta \) - closed mapping.

Proof. Obvious.

Theorem 3.4. A mapping \( f : X \rightarrow Y \) is \( p\theta \) - perfect if and only if \( [\text{ad}_\theta]_Y f(\Phi) \subseteq f(\text{ad}_\Phi) \), for every filter base \( \Phi \) in \( X \).
Proof. Suppose \( f : X \to Y \) is \( p_\theta \)-perfect mapping. Let \( \Phi \) be a filter base in \( X \) and \( y \in \{ \text{ad}_{y_\nu}(\Phi) \} \). Then there exists a filter base \( \Gamma \) in \( f(X) \) which is subordinate to \( f(\Phi) \) and \( p_\theta^* \) converges to \( y \). Put

\[
H = \{ f^{-1}(G) \cap F : G \in \Gamma, F \in \Phi \},
\]

then \( H \) is a filter base in \( X \) subordinate to \( f^{-1}(\Gamma) \). Since \( f \) is \( p_\theta \)-perfect, \( f^{-1}(\Gamma) \) is \( p_\theta^* \)-directed towards \( f^{-1}(y) \). Therefore, we have \( f^{-1}(y) \cap (\{ \text{ad}_{y_\nu}H \}) = \emptyset \) and hence \( y \in \{ \text{ad}_{y_\nu} \} \). Conversely suppose that the condition holds and \( f \) is not \( p_\theta \)-perfect. Then there exists a filter base \( \Phi \) in \( f(X) \) such that \( \Phi \) \( p_\theta^* \) converges to a point \( y \in Y \) and \( f^{-1}(\Phi) \) is not \( p_\theta \)-directed towards \( f^{-1}(y) \). Thus there exists a filter base \( \Gamma \) in \( X \) which is subordinate to \( f^{-1}(\Phi) \) and \( f^{-1}(y) \cap (\{ \text{ad}_{y_\nu} \} \Gamma) = \emptyset \). Therefore, we have \( y \in (\{ \text{ad}_{y_\nu} f(\Gamma) \}) \) and hence \( y \in \text{pcl}_y(\{ f(G) \}) \) for some \( G \in \Gamma \). Then there exists a preopen set \( V \) containing \( y \) such that \( (\text{pdl}(V)) \cap f(\Gamma) = \emptyset \). Since \( \Phi \) \( p_\theta^* \) converges to \( y \) and \( \Gamma \) is subordinate to \( f^{-1}(\Phi) \), there exists \( G \in \Gamma \) such that \( f(G) \subset \text{pcl}(V) \). Consequently, we have \( G \cap G = \emptyset \). This contradicts that \( \Gamma \) is a filter base.

4. \( p \)-closed spaces

Definition 4.1. A space \( X \) is called \( p \)-closed [3] if every cover of \( X \) by preopen sets has a finite subcover whose preclosures cover \( X \).

Definition 4.2. A subset \( A \) of a space \( X \) is called \( p \)-closed relative to \( X \) if for every cover \( \{ V_\alpha : \alpha \in \Delta \} \) of \( A \) by preopen sets of \( X \), there exists a finite subfamily \( \Delta_0 \) of \( \Delta \) such that \( A \subset \bigcup \{ \text{pdl}(V_\alpha) : \alpha \in \Delta_0 \} \).

Theorem 4.1. A subset \( B \) of a space \( X \) is \( p \)-closed relative to \( X \) if and only if \( \emptyset \neq B \cap \bigcap (\{ \text{ad}_{v_\nu} \} \Phi) \neq \emptyset \) for every filter base \( \Phi \) in \( B \).

Proof. Suppose that \( B \) is \( p \)-closed relative to \( X \). Assume that there exists a filter base \( \Phi \) in \( B \) such that \( B \cap \{ \text{ad}_{v_\nu} \} \Phi = \emptyset \). Then for each \( x \in B \) there exists a preopen set \( V_x \) containing \( x \) and an \( F_x \in \Phi \) such that \( F_x \cap \text{pdl}_{v_x}(V_x) = \emptyset \). Since \( B \) is \( p \)-closed relative to \( X \), there exists a finite number of points \( x_1, x_2, \ldots, x_n \) in \( B \) such that

\[
B \subset \bigcup \{ \text{pdl}(V_{x_i}) : i = 1,2,\ldots, n \}.
\]

Put \( F = \{ F_{x_i} : i = 1,2,\ldots, n \} \), then we obtain \( F \cap B = \emptyset \). This contradicts that \( \Phi \) is a filter base.
Conversely suppose $B \cap (\cap_{\Phi} \, \Phi) \neq \emptyset$, for every filter base $\Phi$ in $B$. Assume that $B$ is not $p$-closed relative to $X$. Then there exists a cover $\{ V_\alpha : \alpha \in \Delta \}$ of $B$ by preopen sets of $X$ such that:

$$B \subseteq \bigcup \{ \text{pol}(V_\alpha) : \alpha \in \Delta \},$$

for every $V \in \Gamma(\Delta)$.

where $\Gamma(\Delta)$ denotes the family of all finite subsets of $\Delta$. Now put

$$F_V = \bigcap \{ B - \text{pol}(V_\alpha) : \alpha \in \Delta \},$$

for each $V \in \Gamma(\Delta)$.

Then $\Phi = \{ F_V : V \in \Gamma(\Delta) \}$ is a filter base in $B$ and $B \cap (\cap_{\Phi} \, \Phi) = \emptyset$. This is a contradiction. Therefore $B$ is $p$-closed relative to $X$.

**Theorem 4.2.** If a mapping $f : X \to Y$ is $p\theta$-perfect, then $f^{-1}(B)$ is $p$-closed relative to $X$, for every $p$-closed relative to $Y$ set $B$ of $Y$.

**Proof.** This follows from Theorem 3.4 and Theorem 4.1.

**Theorem 4.3.** A mapping $f : X \to Y$ is $p\theta$-perfect if and only if

(i) $f$ is $p\theta$-closed, and

(ii) $f^{-1}(y)$ is $p$-closed relative to $X$, for each $y \in Y$.

**Proof.** Necessity. Follows from Theorem 3.1 and Theorem 4.2.

Sufficiency. This is proven in a similar manner as the proof of conversely of Theorem 3.4.

**Theorem 4.4.** The composition $gof : X \to Z$ of $p\theta$-perfect mappings $f : X \to Y$ and $g : Y \to Z$ is $p\theta$-perfect mapping.

**Proof.** For $p\theta$-closedness of $(gof)$, this follows from Theorem 3.1 and Theorem 3.3 and

$$(gof)^{-1}(z) = f^{-1}(g^{-1}(z))$$

is $p$-closed follows from Theorem 4.3 and Theorem 4.2.

**Theorem 4.5.** Let $S$ be a singleton with its unique topology. For a space $X$, the following statements are equivalent

(i) $X$ is $p$-closed

(ii) The constant mapping $c : X \to S$ is $p\theta$-perfect.

**Proof.** The equivalence (i) $\iff$ (ii) follows from Theorem 4.3.

**Theorem 4.6.** If $f : X \to Y$ is $p\theta$-perfect mapping and $Y$ is $p$-closed, then $X$ is $p$-closed.

**Proof.** We show that the constant mapping $k : X \to S$ is $p\theta$-perfect. Since $Y$ is $p$-closed, therefore the mapping $c : Y \to S$ is $p\theta$-perfect. Now $k$ is $p\theta$-perfect follows by noting that it is the composition $(c \circ f)$ of two $p\theta$-perfect mappings. Hence $X$ is $p$-closed.
A. S Bin Shalha  Pre-$\theta$-perfect mappings and ....

References

الرّواسب قبل النّامة من النوع 0 وفرعّات $p$ المغلّة

عبدالله سالم بن شحة
قسم الرياضيات - جامعة عدن - اليمن

في هذا البحث قدمنا مفهوم الرواسب قبل النّامة من النوع 0 وفرعّات $p$ المغلّة بعض من خواصها وخصائصها، كما قمنا أيضاً في هذا البحث بتقديم بعض الخصائص لفرعّات $p$ المغلّة.