Delta J. Sci. 35 (2011) 20- 24 MATHEMATICS

Regular preopen L-sets and extremally predisconnected spaces

Abdulla Salem Bin Shahna

Department of Mathematics , University of Aden , Aden , Yemen

(Received: 29-12-2011)

ABSTRACT: In this paper , we introduce the notions of regular preopen L-sets and extremally predisconnectedness in L-topological space and investigate some of their properties and characterizations .

Key words : Preopen L-set , Regular preopen L-set , Extremally predisconnected space .

1. Introduction

In I-topological spaces , the notion of preopen I-sets were introduced and investigated by Bin Shahna [2] . Our primary goal is to introduce and investigate the notions of regular preopen L-sets and extremally predisconnectedness in L-topological spaces and give some of their properties and characterizations .

2. Preliminaries

The category **DMRG** [5] comprises all complete lattices having an order – reversing involution (') or De Morgan complementation , called complete De Morgan lattices (or algebras) , and morphisms preserving arbitrary joins and all involutes and hence arbitrary meets.

We know that the lattice L is in | **DMRG**| . Thus from now on , let $L \in$ | **DMRG** | and satisfies the first infinite distributive law . For every $A \subset L$, the De Morgan laws hold:

$$(\vee A)' = \wedge \{ a' : a \in A \} \text{ and } (\wedge A)' = \vee \{ a' : a \in A \} .$$

The category **SET** comprises all sets and functions between sets . For $X \in$| **SET** | , L^X denotes the set of all mappings from X to L . Then $(L^X, \leq)$ is a complete lattice and has a quasi – complementation (') [4] . That is $L^X \in$ | **DMRG**| . Thus De Morgan laws are inherited by $(L^X, ')$.

A member $\lambda \in L^X$ is called L –set in X . For λ , $\mu \in L^X$, $\lambda \leq \mu$ in L^X if and only if $\lambda(x) \leq \mu(x)$ in L , for each $x \in X$ [4] .

In an obvious way L-topological spaces and L-continuous functions [4] form a topological category and denoted by **L-TOP** [5] .

Definition 2.1 [1]. Let $X \in$ | **L-TOP** | . Then $\lambda \in L^X$ is called

(i) Regular open (resp. regular closed) if $\lambda = Int\, Cl\, \lambda$ (resp. $\lambda = Cl\, Int\, \lambda$);

(ii) Semiopen (resp. semiclosed) if $\lambda \leq Cl\, Int\, \lambda$ (resp. $Int\, Cl\, \lambda \leq \lambda$) .

Definition 2.2 [2]. Let $X \in$ | **L-TOP** | . Then $\mu \in L^X$ is called

(i) Preopen (resp. preclosed) if $\mu \leq Int\, Cl\, \mu$ (resp. $Cl\, Int\, \mu \leq \mu$);

(ii) α - open (resp. α - closed) if $\mu \leq Int\, Cl\, Int\, \mu$ (resp. $Cl\, Int\, Cl\, \mu \leq \mu$).

Definition 2.3 [6]. Let $X \in$ | L-TOP | and let $\lambda \in L^X$. The L-sets

$$pInt\ \lambda = \vee \{ \mu \in L^X : \mu \leq \lambda \text{ and } \mu \text{ is preopen} \}$$

$$pCl\, \lambda = \wedge \{ \mu \in L^X : \lambda \leq \mu \text{ and } \mu \text{ is preclosed} \},$$

are called the preinterior and the preclosure of λ respectively.

Obviously $pInt\ \lambda$ is the greatest preopen L-set contained in λ and $pCl\ \lambda$ is the smallest preclosed L-set containing λ . Also we have $Int\ \lambda \leq pInt\ \lambda \leq \lambda$ and $\lambda \leq pCl\ \lambda \leq Cl\ \lambda$.

Lemma 2.1[6]. *For* $X \in$ | L-TOP | *and let* $\lambda \in L^X$,

(i) $(pInt\ \lambda)' = pCl\ \lambda'$;

(ii) $(pCl\ \lambda)' = pInt\ \lambda'$

Lemma 2.2. *For* $X \in$ | L-TOP | and $\mu \in L^X$,

(i) $Int\ pInt\ \mu = Int\ \mu$, $Cl\ pCl\ \mu = Cl\ \mu$;

(ii) $pInt\ \mu \leq Int\ Cl\ \mu$;

(iii) $Cl\ Int\ \mu \leq pCl\ \mu$.

Proof. It follows from the fact that $pInt\, \mu$ (resp. $pCl\, \mu$) is a preopen (resp. preclosed) L-set in X.

3. Regular preopen L-sets .

Definition 3.1. Let $X \in$ | L-TOP | . Then $\lambda \in L^X$ is called

(i) Regular preopen if $\lambda = pInt\ pCl\ \lambda$;

(ii) Regular preclosed if $\lambda = pCl\ pInt\ \lambda$.

It is clear that a regular preopen (resp. regular preclosed) L-set is a preopen (resp. preclosed) L-set .

Theorem 3.1. *Let* $X \in$ | L-TOP | . *Then* $\lambda \in L^X$ *is regular preopen if and only if* λ' *is regular preclosed* .

Proof. It follows from Lemma 2.1.

Theorem 3.2. *For* $X \in$ | L-TOP | ,

(i) The preclosure of a preopen L-set is a regular preclosed L-set ;

(ii) The preinterior of a preclosed L-set is a regular preopen L-set .

Proof. We prove only (i) . Let λ be a preopen L-set in X . Then $pInt\ pCl\ \lambda \leq pCl\ \lambda$ and $pCl\ pInt\ pCl\ \lambda \leq pCl\ \lambda$. Now λ is a preopen L-set implies that $\lambda \leq pInt\ pCl\ \lambda$ and hence $pCl\ \lambda \leq pCl\ pInt\ pCl\ \lambda$. Thus $pCl\ \lambda$ is a regular preclosed L-set .

Theorem 3.3. *Let* $X \in$ | L-TOP | . *If* $\lambda \in L^X$ *is preclopen , then* λ *is a regular preopen L-set in* X .

Proof. If λ is preclopen L-set in X , then $\lambda = pInt\ \lambda$ and $\lambda = pCl\ \lambda$, therefore we have $\lambda = pInt\ (pCl\ \lambda\ )$. Hence λ is a regular preopen L-set in X .

Theorem 3.4. *Let $X \in$ | L-TOP | and $\lambda \in L^X$ such that $Cl\ \lambda$ (resp. $pCl\ \lambda$) is a preopen L-set in X. Then λ is a preopen L-set in X.*

Proof. Suppose that $Cl\ \lambda$ is a preopen L-set in X. Then we have $\lambda \leq Cl\ \lambda \leq Int\ Cl\,(Cl\ \lambda) = Int\ Cl\ \lambda$. Hence λ is a preopen L-set in X.

Corollary 3.1. *Let $X \in$ | L-TOP | and $\lambda \in L^X$ such that $Cl\ \lambda$ (resp. $pCl\ \lambda$) is a regular preopen L-set in X. Then λ is a preopen L-set in X.*

Theorem 3.5. *Let $X \in$ | L- TOP |. Then $\lambda \in L^X$ is α-open and regular preopen L-set in X if and only if $\lambda = Int\ Cl\ Int\ \lambda$.*

Proof. Necessity . Let λ be an α-open and regular preopen L-set in X, then $\lambda \leq Int\ Cl\ Int\ \lambda$ and $\lambda = pInt\ pCl\ \lambda$. Now

$$\lambda = pInt\ pCL\ \lambda \geq pInt\,(\ Cl\ Int\ \lambda\ ) \geq Int\ Cl\ Int\ \lambda .$$

Hence $\lambda = Int\ Cl\ Int\ \lambda$.

Sufficiency . Let $\lambda = Int\ Cl\ Int\ \lambda$, then obviously λ is an α-open . Now

$$\lambda = Int\ Cl\ Int\ \lambda \leq Int\ pCl\ \lambda \leq pInt\ pCl\ \lambda$$

Also

$$\begin{aligned} pInt\ pCl\ \lambda &\leq Int\ Cl\,(\ pCl\ \lambda) = Int\ Cl\ \lambda \\ &= Int\ Cl\ (\ Int\ Cl\ Int\ \lambda\ ) \\ &\leq Int\ Cl\,(\ Cl\ Int\ \lambda\ ) \\ &= Int\ Cl\ Int\ \lambda = \lambda . \end{aligned}$$

Hence $\lambda = pInt\ pCl\ \lambda$ i.e. λ is a regular preopen L-set .

Theorem 3.6. *Let $X \in$ |L- TOP |. Then $\lambda \in L^X$ is regular preopen and semiopen L-set in X if and only if λ is a regular open L-set in X.*

Proof. Necessity. Suppose that λ is a regular preopen and semiopen L-set in X. Then $\lambda = pInt\ pCl\ \lambda$ and $\lambda \leq Cl\ Int\ \lambda$. So we have

$$\lambda = pInt\ pCl\ \lambda \leq Int\ Cl\,(\ pCl\ \lambda\,) = Int\ Cl\ \lambda .$$

Also

$$\begin{aligned} \lambda = pInt\ pCl\ \lambda &\geq pInt\,(\ Cl\ Int\ \lambda\ ) \\ &\geq Int\,(\ Cl\ Int\ \lambda\,) \\ &= Int\ Cl\,(\ Cl\ Int\ \lambda\,) \\ &\geq Int\ Cl\ \lambda . \end{aligned}$$

Thus $\lambda = Int\ Cl\ \lambda$ implies λ is a regular open L-set .

Sufficiency. Suppose λ is regular open , then $\lambda = Int\ Cl\ \lambda = Int\ Cl\ Int\ \lambda$ and the result follows immediately from Theorem 3.5.

Corollary 3.1. *Let $X \in$ | L-TOP |. Then $\lambda \in L^X$* is regular preopen and semiclosed L-set in X if and only if λ *is a regular closed L-set in X.*

4. Extremally predisconnected spaces .

Definition 4.1. An $X \in$ | L-TOP | is said to be extremally predisconnected if for all preopen L-sets λ in X, $pCl\,(\lambda)$ is a preopen L-set in X.

Theorem 4.1. *Let $X \in$ | L-TOP | be an extremally predisconnected . Then $\lambda \in L^X$ is preclopen if and only if λ is a regular preopen L-set in X .*

Proof. Necessity . It follows from Thoerem 3.3.

Sufficiency . Since X is an extremally predisconnected and λ is a regular preopen L-set in X , then λ is preopen and so $pCl\ (\lambda)$ is a preopen L-set in X . Hence $\lambda = pInt\,(pCl\ \lambda) = pCl\ \lambda$ which implies that λ is a preclosed L-set in X .

Theorem 4.2. *Let $X \in$ | L-TOP | be an extremally predisconnected . For $\lambda \in L^X$, the following statements are equivalent :*

(i) λ *is a preclopen L-set in X ;*

(ii) $\lambda = pCl\,(pInt\ \lambda)$;

(iii) λ' *is a regular preopen L-set in X ;*

(iv) λ *is a regular preopen L-set in X .*

Proof. (i) $\Rightarrow$ (ii) is obvious .

(ii) $\Rightarrow$ (iii) . Let $\lambda = pCl\,(pInt\ \lambda)$. Then $\lambda' = (pCl\ pInt\ \lambda)' = pInt\,(\ pInt\ \lambda\ )' = pInt\ pCl\ \lambda'$. Hence λ' is a regular preopen L-set in X .

(iii) $\Rightarrow$ (iv) . From Theorem 4.1, λ' is a preopen and preclosed L-set in X . Hence λ is a preopen and preclosed L-set in X . Therefore $\lambda = pInt\ pCl\ \lambda$ and hence λ is a regular preopen L-set in X .

(iv) $\Rightarrow$ (i) . It follows from Theorem 4.1.

Theorem 4.3. *Let $X \in$* | **L-TOP** | be *an extremally predisconnected . Then for $\lambda \in L^X$; $Cl\ \lambda$ (resp. $PCl\ \lambda$) is a regular preopen L-set in X if and only if λ is a preopen L-set in X .*

Proof. Necessity. It follows from Theorem 3.4.

Sufficiency. Let λ be a preopen L-set in X. Then Cl λ is a preopen L-set in X and hence preclopen . Thus $Cl\ \lambda$ is a regular preopen L-set in X .

Corollary 4.1. *Let $X \in$ | L-TOP | be an extremally predisconnected . Then for $\lambda \in L^X$, the L-sets $Cl\,(Int\ \lambda)$, $Cl\,(pInt\ \lambda)$, $pCl\,(Int\ \lambda)$ and $pCl\,(pInt\ \lambda)$ are regular preopen L-sets in X .*

Theorem 4.4. *Let $X \in$ | L-TOP | be an extremally predisconnected . If $\lambda \in L^X$ is either a regular open or a regular closed L-set , then λ is a regular preopen L-set in X .*

Proof. If λ is regular open , then $\lambda = Int\ Cl\ \lambda$. Now

$$\lambda = Int\ Cl\ \lambda = Int\ Cl\,(pCl\ \lambda\ ) \geq pInt\ pCl\ \lambda .$$

Also

$$pInt\ pCl\ \lambda \geq pInt\,(Cl\ Int\ \lambda\,) \geq Int\ Cl\,(\ Int\ \lambda\,)$$
$$= Int\ Cl\ \lambda = \lambda .$$

Thus $\lambda = pInt\ pCl\ \lambda$ implies λ is a regular preopen L-set .

Similarly if λ is regular closed , then $\lambda = Cl\ Int\ \lambda$, and we have

$$\lambda = Cl\ Int\ \lambda = pCl\,(\ Cl\ Int\ \lambda\,) \geq pInt\ pCl\,(Cl\ Int\ \lambda\,)$$
$$= pInt\ pCl\ \lambda .$$

Also

$$pInt\ pCl\ \lambda = pInt\ pCl\,(\ Cl\ Int\ \lambda\ ) \geq pInt\ pCl\,(\ Int\ \lambda\,)$$
$$= pCl\,(Int\ \lambda\,) \geq Cl\ Int\,(\ Int\ \lambda\,)$$

$$= Cl\, Int\ \lambda = \lambda\ .$$

Hence $\lambda = pInt\, pCl\ \lambda$ and λ is a regular preopen L-set in X.

References

[1] K.K.Azad , On fuzzy semicontinuity , fuzzy almost continuity and fuzzy weakly continuity , J.Math. Anal. Appl. 82 (1981) , 714-732.
[2] A.S. Bin Shahna , On fuzzy strong semicontinuity and fuzzy precontinuity , Fuzzy Sets and Systems 44 (1991) , 303-308 .
[3] C.L. Chang , Fuzzy topological spaces , J. Math. Anal. Appl. 24 (1968) , 182-190 .
[4] T. Kubiak , Separation axioms : extention of mappings and embedding of spaces , in : U. Höhle , S.E. Rodabaugh (Eds.) , Mathematics of Fuzzy Sets : Logic , Topology and Measure Theory , The Handbooks of Fuzzy Set Series , Vol. 3 , Kluwer Academic Publishers , Dordrecht , 1999, pp. 433-479 .
[5] S.E. Rodabough , Categorical foundations of variable- basis fuzzy topology , in : U. Höhle , S.E. Rodabaugh (Eds.) , Mathematics of Fuzzy Sets : Logic , Topology and Measure Theory , The Handbooks of Fuzzy Set Series , Vol. 3 , Kluwer Academic Publishers , Dordrecht , 1999, pp. 273-388 .
[6] M. K. Singal and N. Prakash, Fuzzy preopen sets and fuzzy preseparation axioms, Fuzzy Sets and Systems 44 (1991) , 273-281 .

المجموعات المنتظمة قبل المفتوحة -L والفراغات غير المترابطة القبلية المتطرفة

عبدالله سالم بن شحنة
قسم الرياضيات- جامعه عدن- اليمن

في هذا البحث قمنا بتقديم مفهوم المجموعات قبل المفتوحة المنتظمة الفازية -L ومفهوم غير الترابط القبلي المتطرف في الفضاءات الفازية -L كما قمنا بدراسة بعض خواص هذين المفهومين وبعض خصائصهما .