Delta J. Sci. 35 (2011)1-9 MATHEMATICS

Asymptotic Behaviour Functional Differential Systems

Dr. Fatma M.Kandil

Mthematics Department ,Faculty of Science
King Abdul Aziz University
Jeddah - Saudi Arabia
(Received 29-3-2011)

Abstract: A functional differential system is considered. The existence and stability of the solution are investigated.

A.M.S. Classifications: 34K20, 34K17.

Keywords: Functional, differential, stability.

1. Introduction

A more general type of differential equation, is one in which the unknown function occurs with various different arguments. The simplest and perhaps most natural type of functional differential equation is a delay differential equation. The existence, uniqueness and stability are discussed in specialized books [2, 3, 4, 6] and papers [1, 5, 7] for example.

Let R^n denote the n-dimentional real Euclidean space for a given $\tau \geq 0$. Let $\wp^n = C[-\tau,0],R^n)$ denote the space of continuous functions with domain $[-\tau,0]$ and range in R^n. For any element $\phi \in \wp^n$, we define the norm

$$\|\phi\|_0 = \max_{-\tau \leq s \leq 0} \|\phi(s)\|$$

where $\|\cdot\|$ is any convient norm in R^n. Suppose that $x \in C[(-r,\infty),R^n]$. For any $t \geq 0$, let x_t denote the element of $\wp^n$ defined by

$$x_t(s) = x(t+s), \qquad -\tau \leq s \leq 0$$

Let $C(\rho)=\{\phi\in \wp^n : \|\phi\|_0 < \rho\}$ where ρ is a given constant.

Now we consider the functional differential system

$$x' = f(t, x_t) \quad , \quad x_{t_0} = \phi_0 \tag{1.1}$$

where $f \in C[\Im \times C(\rho), R^n]$, we shall assume that $f(t,0)\equiv 0$ and $f(t,\phi)$ is smooth enough to guarantee the existence of solutions (1.1) in the future. For the definition of stability of solution see [1, 2, 4, 7].

2. Main Results

We shall state a very general set of conditions for prenventing the solutions that start in a given set of R^n through any given part of its boundary.

Now we shall make use of the following theorem:

Theorem A [6, Theorem 6.9, pp 37-38] . Let $m \in C[[t_0 - \tau, \infty), R_+]$, and satisfy the inequality

$$D\_m(t) \le f(t, m(t), m_t) \ , \quad t > t_0$$

where $f \in [\Im \times R_+ \times \wp_+, R]$. Assume that $f(t,x,\phi)$ is nondecreasing in ϕ for each (t,x) and that $r(t_0,\phi_0)$, $\phi_0 \in \wp_+$, is the maximal solution of the equation

$$x' = f(t, x, x_t)$$

existing for $t \ge t_0$. Them $m_{t_0} \le \phi_0$ implies

$$m(t) \le r(t_0, \phi_0)(t) \ , \quad t \ge t_0 \ .$$

Theorem 1: Let H and E be open subsets of R^n, such that $\overline{H} \subset E$ and let $G \subset \partial H$ (where ∂H , the boundary of H). Let $V \in C[[-\tau,\infty)_x E, R]$, $a \in C[[-\tau,\infty), R]$ and $g \in C[\Im \times R \times \wp, R]$, where $V(t,x)$ is locally

Lipschitzian in x and $g(t,u,v)$ is nondecreasing in v for each $(t,u)\in \Im\times R$

.

Assume that

A$_1$) $\phi_0(s)\in H$, for $-\tau\le s\le 0$ where $\phi_0\in \wp^n$,

A$_2$) $V_{t_0}<a_{t_0}$ where $V_{t_0}=V(t_0+s,\phi_0(s))$, $-\tau\le s\le 0$

A$_3$) $V(t,x)\ge a(t),(t,x)\in \Im\times G$

A$_4$) $D^+V(t,\phi(0),\phi)\le g(t,V(\phi,0),V_t)$, $(t,\phi(s))\in \Im\times H$, $-\tau\le s\le 0$;

where $D^+V(t,\phi(0),\phi)=\lim_{k\to 0}\sup h^{-1}[V(t+h,\phi(0)+hf(t,\phi)-V(t,\phi(0))]$

A$_5$) any solution $u(t_0,\sigma_0)$ of the functional differential equation

$$u'=g(t,u,u_t)\ ,\ u_{t_0}=\sigma_0<a_{t_0} \qquad (2)$$

satisfies condition

$$u=(t_0,\sigma_0)(t)<a(t)\ ,\ t\ge t_0$$

Then there exists no $t^*>t_0$ such that

$$x(t_0,\phi_0)(t)\in H,\ t_0<t<t^*,\ \text{and}\ \phi(t_0,\phi_0)(t^*)\in G.$$

Proof : Suppose there exists $t^*>t_0$ satisfying

$$x(t_0,\phi_0)(t)\in H,\ t\in[t_0,t^*),\ \text{and}\ x(t_0,\phi_0)(t^*)\in G.$$

From assumption A$_3$, it follows that

$$V(t^*,x(t_0,\phi_0)(t^*)\ge a(t^*) \qquad (3)$$

Let $\qquad m(t)=V(t,x(t_0,\phi_0)(t))\ ,\ t_0\le t<t^*$

and using (A$_4$) with Lipschitzian character of V, we get

$$D^+m(t)\le g(t,m(t),m_t)\ ,\ t_0\le t<t^* \qquad (4)$$

Furthermore from (A$_2$) we have

$$m_{t_0}<a_{t_0} \qquad (5)$$

Now, by application Theorem A we have

$$V(t, x(t_0, \phi_0)(t)) \le r(t_0, \sigma_0) \quad , \quad t \in [t_0, t^*) \qquad (6)$$

where $r(t_0, \sigma_0)$ is the maximal solution of (2). This together with V gives

$$V(t^*, x(t_0, \phi_0)(t^*) < a(t^*) \, ,$$

which contradicts (3). This completes the Proof.

Remark : If all the assumptions of Theorem 1 hold except that (V) is replaced by $D^* a(t) > g(t, a, a_t)$ for $t \in \Im$, then the conclusion of Theorem 1 remains the same .

Theorem 2: Let E be an open subset $R^n, F \subset E$, $D \subset E$ with $\overline{D} \subset E$.

Assume that:

B1) $V \in C[[-r, \infty) \times E, R]$, $V(t, x)$ is locally Lipsehzian in x ;

B2) $g \in C[\Im \times R \times \wp, R]$, $g(t, u, v)$ is nondecreasing in v for each $(t, u) \in \Im \times R$ and

$$D^+ V(t, \phi(0), \phi) \le g(t, V(t, \phi(0)), V_t) \text{ and } (t, \phi) \in \Im \times \wp^n$$

with $\phi(s) \in E, -\tau \le s \le 0$;

B3) $\phi_0 \in F, -\tau \le s \le 0 \Rightarrow x(t_0, \phi_0)(t) \in E, t \ge t_0$;

B4) $(t, x) \in \Im \times E \setminus D$ implies $V(t, x) \ge a(t)$, $a \in C[\Im, R]$,

B5) There exists a $T^* = T^*(t_0, \sigma_0)$ such that for any solution $u(t_0, \sigma_0)$ of the functional differential equation

$$u' = g(t, u, u_t) \quad , \quad u_{t_0} = \sigma_0$$

Satisfies the relation

$$u(t_0, \sigma_0)(t) < a(t) \quad , \quad t \ge t_0 + T^\infty \text{ holds.}$$

Then there exists a $T = T(t_0, \phi_0) > 0$ such that

$$x(t_0, \phi_0)(t) \in D \text{ for all } t \ge t_0 + T \, .$$

Proof : Let $(t_0,\phi_0(s))\in \Im\times F$, for $-\tau\le s\le 0$, so that the assumption (B₃) $x(t_0,\phi_0)(t)\in E$, for all $t\ge t_0$. Putting

$$\sigma_0(s)=V(t_0+s,\phi_0(s))=V_{t_0}(s)\ ,\quad -\tau\le s\le 0\ . \tag{7}$$

Now we define

$$T(t_0,\phi_0)=T^*(t_0,V_{t_0}).$$

We clain that $x(t_0,\phi_0)(t)\in D$, for all $t\ge t_0+T$, otherwise there exists a sequence $\{t_k\}$ such that $t_k\ge t_0+T$, $t_k\to\infty$ as $k\to\infty$ and $x(t_0,\phi_0)(t_k)\in E\setminus D$.

Then by assumption (B_4) we have

$$V(t_k,x(t_0,\phi_0)(t_k)\ge a(t_k)\ ,\qquad k=1,2,.... \tag{8}$$

Furthermore, in view of the assumption B_1 , B_2 , B_3 , (7) and Theorem A in [6], we conclude that

$$V(t_k,x(t_0,\phi_0)(t_k)<a(t_k)\ ,\qquad t_k\ge t_0+T \tag{9}$$

This contradiction proves the result.

Theorem 3: Assume that

(C_1) $V\in C[[-\tau,\infty)\times S(\rho)\setminus\{0\}\ ,R]$, $V(t,x)$ is locally Lipschitzian in x and $V(t,x)\to-\infty$ as $\|x\|\to 0$, for each $t\in[-\tau,\infty)$;

(C_2) $b\in C[[-\tau,\infty)\times(0,\rho),R]$ and for $(t,x)\in\Im\times S(\rho)\setminus\{0\}$,

$V(t,x)\ge b(t,\|x\|)$;

(C_3) $g\in C[\Im\times R\times\wp,R]$, $g(t,u,v)$ is no decreasing in v for each $(t,u)\in\Im\times R$ and for $(t,\phi)\in\Im\times C(\rho)\setminus\{0\}$,

$D^+V(t,\phi(0),\phi)\le g(t,V(t,\phi(0)),V_t)$;

(C_4) any solution $u(t_0,\sigma_0)$ of the functional differential equation

$u'=g(t,y,u_t)\ ,\ u_{t_0}(s)=\sigma_0(t)<b_{t_0}(s,r)$

for every $r \in (0,\rho)$ and $s \in [-\tau,0]$, satisfies

$$u(t_0,\sigma_0)(t) < b(t,r) \ , \quad t \geq t_0 \quad \text{for every } r \in (0,\rho) .$$

Then the trivial solution of (1.1) is equistable.

Proof: In view of assumption (C_1), for every $(t_0+\theta,\varepsilon) \in [t_0-\tau,\infty) \times (0,\rho)$ there exists *a* $\delta^*_\theta = \delta^*_\theta(t+\theta,\varepsilon)$ such that $\phi_0(\theta) \in S(\delta^*_\theta) \setminus \{0\}$ implies

$$V(t_0+\theta,\phi_0(\theta)) < b(t_0+\theta,\varepsilon), \text{ for } \theta \in [-\tau,0]$$

Our aim is to choose δ which is independent of $\theta \in [-\tau,0]$. For this purpose, we notice that the continuity of V, b and ϕ_0 together with the fact that $S(\delta^*_\theta) \setminus \{0\}$ is open set, implies that there exists $\eta_\theta > 0$ such that

$V(t_0+s,\phi_0(s)) < b(t_0+s,\varepsilon)$, holds for $s \in (-\eta_\theta,\eta_\theta) \cap [-\tau,0]$ and

$\phi_0(s) \in S(\delta^*_\theta) \setminus \{0\}$.

Such a choice of neighbourhoods is possible for all $\theta \in [-\tau,0]$.

Consider the collection of open sets of $[-\tau,0]$ defined by

$$U = \{U_\theta : U_\theta = (\ -\eta_\theta,\eta_\theta) \cap [-\tau,0] , \text{ for all } \theta \in [-\tau,0]\}.$$

It is easy to verify that it forms an open covering for $[-\tau,0]$. Since this set is compact by Heine-Borel Theorem [9 pp 42], we can extract a finite subcover corresponding to $\eta_{\theta_1},\eta_{\theta_2},\eta_{\theta_3},\ldots\eta_{\theta_n}$, some fixed integer n . Consider the corresponding numbers

$$\delta^*(t+\theta_1,\varepsilon),\delta^*(t_0+\theta_2,\varepsilon),\ldots\delta^*(t_0+\theta_n,\varepsilon) \ ,$$

and set

$$\delta = \min\{\delta^*(t_0+\theta_1,\varepsilon),\delta^*(t_0+\theta_2,\varepsilon),\ldots\delta^*(t_0+\theta_n,\varepsilon)\} .$$

Then for $\theta \in [-\tau,0]$, we have

$$\phi_0(0) \in S(\delta) \setminus \{0\} \ \text{ and } \ V(t_0+\theta,\phi_0(\theta)) < b(t_0+\theta,\varepsilon)$$

or $\qquad V_{t_0} < b_{t_0}(\varepsilon)$

whenevre $\quad \phi_0 \in C(\delta) \setminus \{0\}$.

Setting now $E = S(\rho) \setminus \{0\}, H = S(\varepsilon) \setminus \{0\}, G = \partial S\{\varepsilon\}$ and $a(t) = b(t,\varepsilon)$, we see that all the hypotheses of Theorem 3 are verified. Hence the conclusion follows.

Remark: Notice that the Liapunov-like function used in this theorem is neither positive definite nor defined at $x = 0$.

Theorem 4: Suppose that the hypotheses of Theorem 3 hold. Assume further that $b(t,r)$ is nondecreasing in r for each $t \in \Im$ and that there exists *o* $T^* = T^*(t_0, \sigma_0) > 0$ such that every solution $u(t_0, \sigma_0)$ of

$$u' = g(t,u,u_t) \ , \quad u_{t_0} = \sigma_0$$

satisfies the relation

$$u(t_0,\sigma_0)(t) < b(t,r) \ , \quad t \geq t_0 + T^*$$

for all $r \in (0,\rho)$. Then the trivial solution of (1) is equiasymptotically stable.

Proof: Since by Theorem 3, the trivial solution of (1) is equistble, for $\varepsilon = \rho$, *a* $\delta_0 = \delta(t_0, \rho)$ such that $\phi_0 \in C(\delta_0) \setminus \{0\}$ implies

$$x(t_0,\phi_0)(t) \in S(\rho) \setminus \{0\} \ , \quad t \geq t_0$$

Set $F = S(\delta_0) \setminus \{0\}$ and $E = S(\rho) \setminus \{0\}$. Then the hypothesis (B_3) of Theorem 2 is verified. Let $(t_0, \varepsilon) \in \Im \times (0,\rho)$, and set $D = S(\varepsilon) \setminus \{0\}$. Then for $(t,x) \in \Im \times E \setminus D$ and because of the hypothesis (C_2) of Theorem 3, together with monotonicity of $b(t,r)$ we have

$$V(t,x) \geq b(t,\varepsilon) \ , \quad \text{for} \quad (t,x) \in \Im \times E \setminus D$$

Choosing $a(t)=b(t,\varepsilon)$, we see that B_4 of Theorem 2 is verified. The rest of the hypotheses were checked already in the proof of Theorem 3, Hence the conclusion of the theorem follows from Theorem 2.

Remark: Observe that the Liapunov function used in Theorem 4 need not to be positive definite , decrescent and its derivative need not to be negative definite. Moreover it is not defined at $x=0$.

References

[1] Berezansky, L. Braverman, E., On stability of some linear and nonlinear delay differential equations. J. Math. Anal. Appl. Vol. 314 (2006) 391-411.

[2] Diver, R. Ordinary and Delay Differential equation springer-Verlag, New York, 1977.

[3] Gyori, I. and Ladaas G., Oscillation Theory of Delay Differential Equations, Clarenton Press, Oxford, 1991.

[4] Hale, J. Theory of functional differential equation 1973.

[5] Ladas, G. Sficas, Y, Stavroulakis, I., Asymptotic behavior of solutions of retarded differential equations Proc. Amer. Math. Soc. Vol. 88, (1983) 247-253.

[6] Lakshmikantham, V. and Leela, S., Differential and integral inequalities, Vol. I and II, Academic Pres, New York 1969.

[7] Partheniadis, E. Stability and oscillation of neutral delay differential equations. Differential and Intrgral Equations Vol.1, (1988), 459-472.

[8] Rouche, N, Habets, P. and Laloy, M., Stability Theory by Liapunov's Direct Methods. Springer-Verlage New York, 1977.

[9] Royden, H. Real Analysis, The Macmillan Comp. New York, 2nd End 1968.

Delta J. Sci. 35(2011) 1-9

السلوك القربى لنظام تقاربى دالى

فاطمه محمد قنديل
قسم الرياضيات-كليه العلوم- جامعه الملك عبد العزيز- جده- المملكه العربيه السعوديه

تعتبر المعادلات التفاضلية ذات تأخير أكثر عمومية من المعادلات التفاضلية العادية ويناقش هذا البحث

دراسة نظام تفاضلى دالى فى R^n من هذا النوع وهذا النظام متعادل على الصورة

$$x' = f(t, x_t) \quad , \quad x_{t_0} = \phi_0$$ حيث $f \in C[\Im \times C(\rho), R^n]$

$$x_t(s) = x(t+s) \; , \qquad -\tau \le s \le 0$$

واستنتاج شروطا كافية على النظام ليضمن وجود الحلول ووحدانيتة ودراسة استقرار النظام