



## SIMPLE APPROACH OF THE SERVER'S BUSY PERIOD FOR SELF-SERVICE SYSTEM WITH FEEDBACK

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### KEY WORDS

Busy period, Self-service queue, Feedback, Probability density function, Distribution function, Probability generating function.

### ABSTRACT

In this paper, we provide an easy method to obtain the distribution of busy period for self-service queue with feedback. Using probability generating function we obtain the probability that there are  $n$  units in the system at time  $t$ ,  $p_n(t)$  and the probability density function and the busy period distribution function. Concept of busy period self-service queue without feedback has been derived and discussed as special case of proposed model. Some numerical values of arrive rate  $\lambda$ , service rate  $\mu$ , feedback  $q$  and time  $t$  are given showing the effect of correlation between the feedback (the time) and the probability density function (distribution function) of busy period.

### 1. Introduction

The busy period analysis has been studied by many researchers. The pioneer work began to appear by Borel [1] and has now become one of the salient features of queueing theory. Kotb and Al-seedy [11] provide a simple method to obtain the distribution of busy period for queue:  $M/M/1$  with state-dependent service rate. Bunday and EL-Badri, [2] gave only the equations of the busy period for the queue:  $M/M/1$ . Gross and Harris [6] and Medhi [13] studied the busy period distribution for the queue:  $M/M/1$  using Laplace transform of the generating function but without any additional concepts. Some increasing results on  $M/G/\infty$  queue

system busy cycle distribution are presented by Ferreira, et. al. [3]. Some explicit formulas for the distribution of the number of customers served during a busy period are derived by Stadge [15]. Tackacs [16] studied the busy cycle, a busy period followed by an idle period, for the  $M/G/\infty$  queue. Other related studies are presented by Hadidi [7], Hall [8], Ramalhoto and Ferreira [14], Jain and Reiss [9], Kumar, et. al. [12], Ferreira and Alberto [3], [4], Ferreira [5], and Kalidass, et. al. [10].

The aim of this paper is to study the transient behavior of the server's busy period for self-service model with feedback concept.

The probability that there are  $n$  units in the system at time  $t$ , the probability density function and the busy period distribution function are obtained using probability generating function and Lagrange's equation. Busy period self-service queue without feedback concept has been derived and discussed as particular case of this model. Finally, some numerical values of arrive rate  $\lambda$ , service rate  $\mu$ , feedback  $q$  and time  $t$  are given showing the effect of correlation between the feedback (the time) concept and the probability density function (distribution function) of busy period.

## 2. NOTATIONS AND ASSUMPTIONS

We define the following parameters to build the system of this model:

$p_n(t) \equiv$  Transient-state probability that there are  $n$  units in the system at time  $t$ , both waiting and in service.

$p_0(t) \equiv$  Probability that there are no units in the system at time  $t$ .

$n \equiv$  Number of units in the system,  $n \geq 0$ .

$H(s, t) \equiv$  Probability generating function.

$T \equiv$  Random variable represents the duration of busy period.

$B(t) \equiv$  Distribution function for busy period.

$b(t) \equiv$  Probability density function for busy period.

The assumptions for this model are listed as follows:

(1) Customers arrive at the server one by one according to Poisson process with rate  $\lambda (> 0)$  and mean inter-arrival time is  $1/\lambda$ .

(2) Service times of customers are follow the exponential distribution with rate  $\mu (> 0)$  and mean service time is  $1/\mu$ ,  $0 < \lambda < \mu$ .

(3) Customers are served according to first-in first-service (**FIFO**) discipline.

(4) After completion of each service the customer departure the system with probability  $p$  or joins at the end of the original queue as feedback customer with probability  $q = 1 - p$ .

(5) A busy period (**BP**) begins when a customer arrives at an idle channel and ends when the channel next becomes idle.

(6) The initial condition will be  $p_1(0) = 1$  and  $p_n(0) = 0$ ,  $n \neq 1$  since the busy period starts with one in the system.

## 3. MODEL DESCRIPTION AND ANALYSIS

Consider unlimited servers queueing system:  $M/M/\infty$  with Poisson arrivals and exponential service times where customers arrive at rate  $\lambda_n = \lambda$ . Assume that the customers serve themselves at rate,  $\mu_n = n\mu$ . Consider the duration of the busy period to be a random variable  $T$ , with probability density function  $b(t)$  and distribution function  $B(t)$ , then the probability that the system has reached the state zero by the time  $t$  is:

$$p_0(t) = p(T \leq t)$$

(1)

then:

$$B(t) = p_0(t)$$

(2)

also:

$$b(t) = \frac{dB(t)}{dt} = p_0'(t) = q\mu p_1(t)$$

(3)

From the above notations and assumptions and applying Markov conditions, there are three states, thus the transition probability differential-difference equations are:

$$p_0'(t) = q\mu p_1(t),$$

$$\begin{matrix} n & = & 0 \end{matrix}$$

(4)

$$p_1'(t) = -(\lambda + q\mu) p_1(t) + 2q\mu p_2(t)$$

$$\dots$$

$$p_n'(t) = -(\lambda + nq\mu) p_n(t) + \lambda p_{n-1}(t) + (n+1)q\mu p_{n+1}(t)$$

$$\dots$$

$$p_{n+1}'(t) = -(\lambda + (n+1)q\mu) p_{n+1}(t) + \lambda p_n(t) + (n+2)q\mu p_{n+2}(t)$$

(5)

$$p_n'(t) = -(\lambda + nq\mu) p_n(t) + \lambda p_{n-1}(t) + (n+1)q\mu p_{n+1}(t)$$

$$\dots$$

$$p_{n+1}'(t) = -(\lambda + (n+1)q\mu) p_{n+1}(t) + \lambda p_n(t) + (n+2)q\mu p_{n+2}(t)$$

(6)

Let us defined the probability generating function as:

$$H(s,t) = \sum_{n=0}^{\infty} p_n(t) s^n, \quad |s| \leq 1$$

(7)

Differentiating equation (7) with respect to  $t$  and use equations (4) - (6), we get:

$$\frac{\partial H(s,t)}{\partial t} = \sum_{n=0}^{\infty} p_n'(t) s^n$$

$$= p_0'(t) + p_1'(t)s + \sum_{n=2}^{\infty} p_n'(t)s^n$$

$$= q\mu p_1(t) + -(\lambda + q\mu)p_1(t)s + 2q\mu p_2(t)s - \lambda \sum_{n=2}^{\infty} p_n(t) s^n$$

$$- q\mu \sum_{n=2}^{\infty} n p_n(t) s^n + \lambda \sum_{n=2}^{\infty} p_{n-1}(t) s^n$$

$$+ q\mu \sum_{n=2}^{\infty} (n+1) p_{n+1}(t) s^n$$

$$= -\lambda \sum_{n=1}^{\infty} p_n(t) s^n - q\mu s \sum_{n=1}^{\infty} n p_n(t) s^{n-1}$$

$$+ \lambda s \sum_{n=2}^{\infty} p_{n-1}(t) s^{n-1}$$

$$+ q\mu \sum_{n=0}^{\infty} (n+1) p_{n+1}(t) s^n$$

$$= -\lambda(1-s)H(s,t) + q\mu(1-s) \frac{\partial H(s,t)}{\partial s}$$

(8)

thus,  $H(s,t)$  satisfies the partial differential equation:

$$\frac{\partial H(s,t)}{\partial t} - q\mu(1-s) \frac{\partial H(s,t)}{\partial s} = -\lambda(1-s)H(s,t)$$

(9)

Our problem is to solve equation (9) for  $H(s,t)$ , subject to the condition that  $H(s, 0) = s$ .

equation (9) is a particular case of what is known as Lagrange's equation (LE).

For the equation:

$$P \frac{\partial H(s,t)}{\partial t} + Q \frac{\partial H(s,t)}{\partial s} = R$$

where  $P$ ,  $Q$  and  $R$  are functions of  $H$ ,  $s$  and  $t$ , we form the related equations:

$$\frac{dt}{P} = \frac{ds}{Q} = \frac{dH(s,t)}{R}$$

Thus for equation (9), the related equations are:

$$\frac{dt}{1} = \frac{ds}{-\mu q(1-s)} = \frac{dH(s,t)}{-\lambda(1-s)H(s,t)}$$

(10)

then:

$$\frac{dt}{1} = \frac{ds}{-\mu q(1-s)}$$

gives:

$$U(s,t,H) = (1-s)e^{-q\mu t} = C_1$$

and:

$$\frac{ds}{-\mu q(1-s)} = \frac{dH(s,t)}{-\lambda(1-s)H(s,t)}$$

gives:

$$V(s,t,H) = H(s,t)e^{-(\lambda/q\mu)s} = C_2$$

where  $C_1$  and  $C_2$  are constant values.

hence:

$$H(s,t) = e^{(\lambda/q\mu)s} \cdot g \left\{ (1-s)e^{-q\mu t} \right\}$$

(11)

where  $g(\cdot)$  is an arbitrary function.

thus, the general solution of equation (9) is given by:

$$V = g(U)$$

Now, from the initial condition, when  $t = 0$ ,

$$H(s, 0) = s = e^{(\lambda/q\mu)s} \cdot g(1-s) \quad (12)$$

then, with  $y = (1-s)$ , i. e.  $s = 1-y$

thus:

$$g(y) = (1-y)e^{-(\lambda/q\mu)(1-y)}$$

and this gives the particular form that  $g(\cdot)$  must have to satisfy the initial condition. We

need  $g\{(1-s)e^{-q\mu t}\}$

hence:

$$H(s, t) = \exp \left\{ - \left( \frac{\lambda}{q\mu} \right) (1-s) \left( 1 - e^{-q\mu t} \right) \right\} \cdot \left\{ 1 - (1-s) e^{-q\mu t} \right\}$$

Using Leibniz's result for the  $n^{\text{th}}$  derivative of a product, we get the probability that there are  $n$  units in the system at time  $t$ :

$$\begin{aligned} p_n(t) &= \frac{1}{n!} \left. \frac{\partial^n H(s, t)}{\partial s^n} \right|_{s=0} \\ &= \frac{e^{-(\lambda/q\mu)(1-e^{-q\mu t})}}{n!} \left[ \left( \frac{\lambda}{q\mu} \right) (1-e^{-q\mu t}) \right]^{n-1} \\ &\quad \cdot \left[ \left( \frac{\lambda}{q\mu} \right) (1-e^{-q\mu t})^2 + n e^{-q\mu t} \right] \end{aligned} \quad (13)$$

The probability density function of the busy period is given by:

$$b(t) = p_0'(t) = q\mu p_1(t)$$

From equation (13) for  $n=1$ , we get:

$$b(t) = e^{-\left(\frac{\lambda}{q\mu}\right)(1-e^{-q\mu t})} \cdot \left[ \lambda (1-e^{-q\mu t})^2 + q\mu e^{-q\mu t} \right] \quad (14)$$

Thus, the distribution function of busy period  $B(t)$ , is given by:

$$B(t) = p_0(t) = e^{-\left(\frac{\lambda}{q\mu}\right)(1-e^{-q\mu t})} \cdot \left( 1 - e^{-q\mu t} \right) \quad (15)$$

In the case of  $q = 1$ , we get the busy period self-service queue:  $M/M/\infty$  without feedback concept, then the probability density function of busy period reduces to:

$$b(t) = e^{-(\lambda/\mu)(1-e^{-\mu t})} \cdot \left[ \lambda (1-e^{-\mu t})^2 + \mu e^{-\mu t} \right] \quad (16)$$

Also, the distribution function of busy period is:

$$B(t) = e^{-(\lambda/\mu)(1-e^{-\mu t})} \cdot \left( 1 - e^{-\mu t} \right) \quad (17)$$

Which are the same results as Gross and Harris [3].

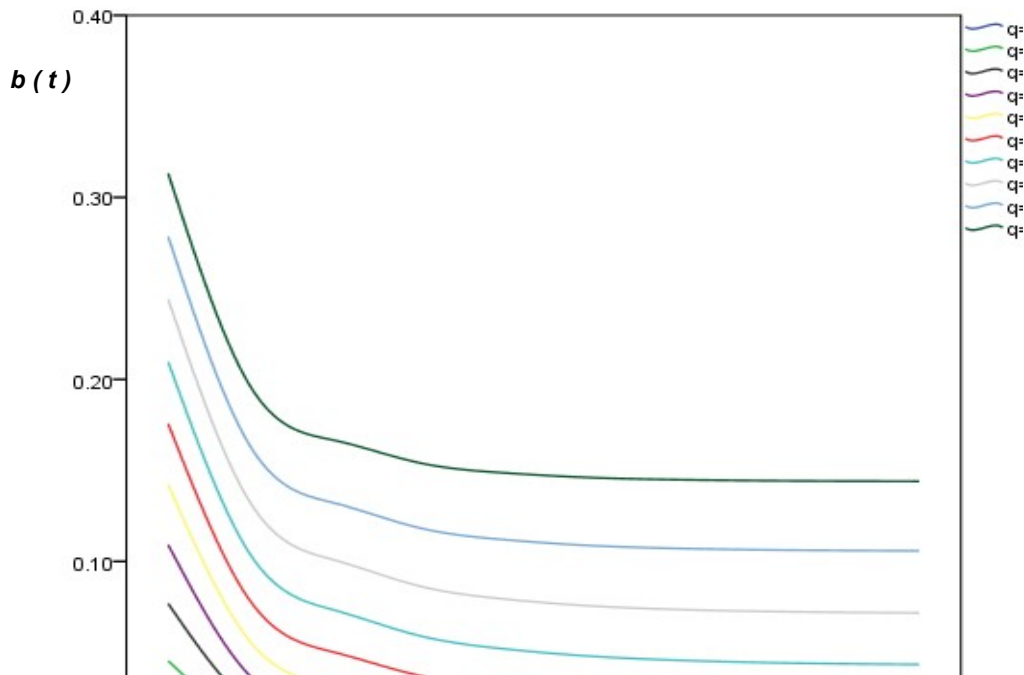
#### 4. SIMULATION

In this section, we present some numerical examples to investigate the relationship between the time  $t$ , and feedback concept  $q$  against probability density function  $b(t)$  and distribution function of busy period  $B(t)$ , for  $\lambda = 2$ ,  $\mu = 4$ . Substituting by  $\lambda$  and  $\mu$  in relations (14) and (15), then we get TABLES 4.1 and 4.2 respectively for some different values of  $t$  and  $q$ :

TABLE 4. 1 The probability density function  $b(t)$  against  $t$  and  $q$ 

$q$	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19
$t$	$b(t)$									
.5	.0148592	.0453852	.0768671	.1091653	.1421475	.1756884	.2096698	.2439807	.2785169	.3131813
1.5	.0024173	.0099217	.0211557	.0361870	.0549416	.0772267	.1027556	.1311745	.1620875	.1950785
2.5	.0004660	.0029701	.0084257	.0177222	.0314754	.0499630	.0731298	.1006404	.1319578	.1664264
3.5	.0001003	.0010343	.0039472	.0102181	.0210916	.0373741	.0593024	.0865829	.1185340	.1542591
4.5	.0000232	.0003992	.0020796	.0066293	.0157625	.0307518	.0520847	.0794680	.1120542	.1487234
5.5	.0000057	.0001684	.0012119	.0047316	.0127833	.0270227	.0481191	.0757327	.1088475	.1461636
6.5	.0000014	.0000771	.0007711	.0036460	.0110114	.0248259	.0458723	.0737329	.1072422	.1449726
7.5	.0000004	.0000381	.0005292	.0029847	.0099083	.0234917	.0445749	.0726504	.1064338	.1444170
8.5	.0000001	.0000203	.0003872	.0025613	.0091985	.0226647	.0438169	.0720608	.1060255	.1441574
9.5	.0000000	.0000115	.0002991	.0022794	.0087306	.0221451	.0433710	.0717386	.1058190	.1440361

Solution of the probability density function for busy period of this model may determine more readily by plotting  $b(t)$  against  $t$  and  $q$  as given in FIGURES 4.1 and 4.2 respectively.



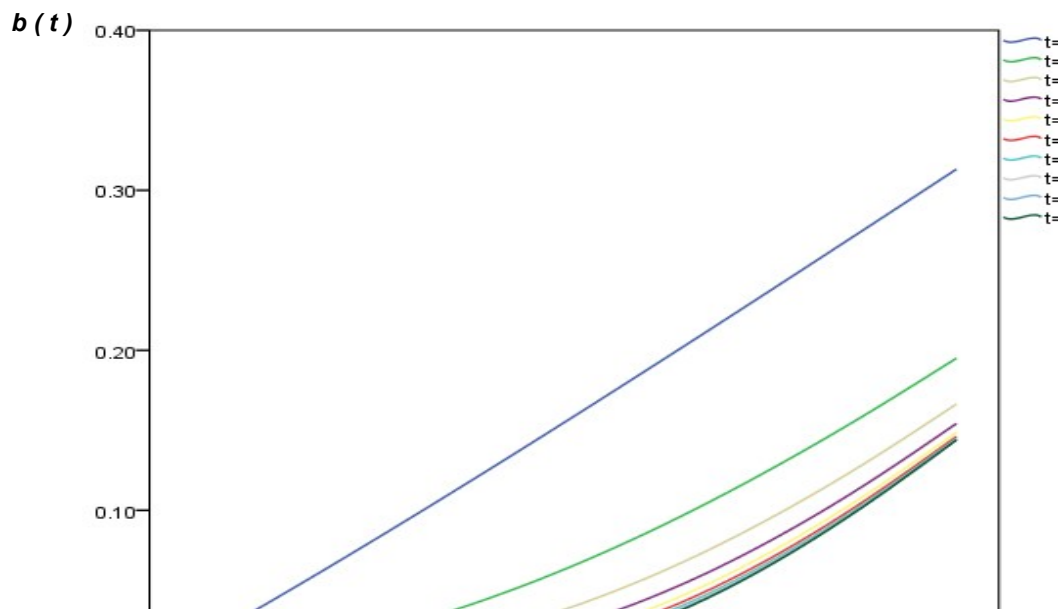
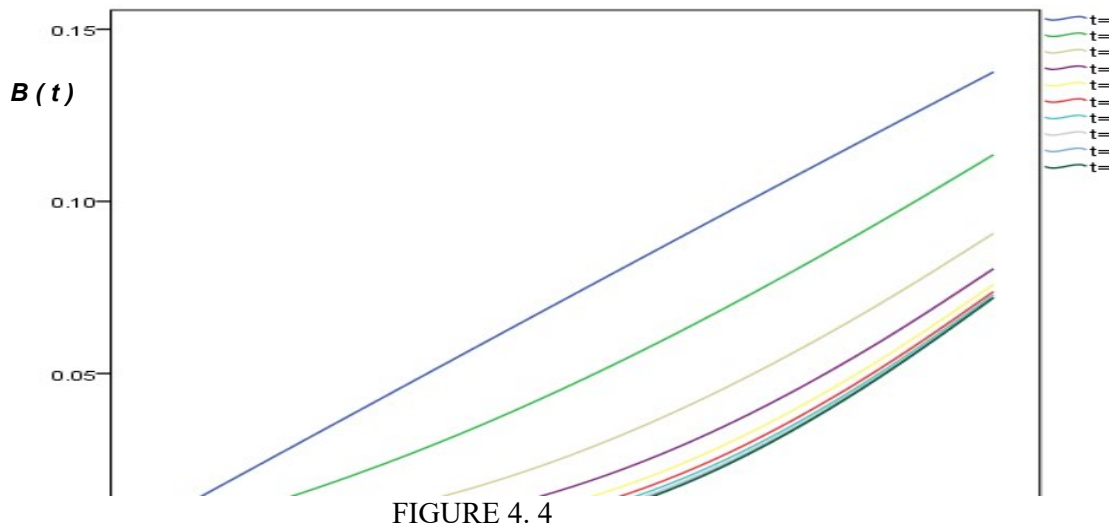
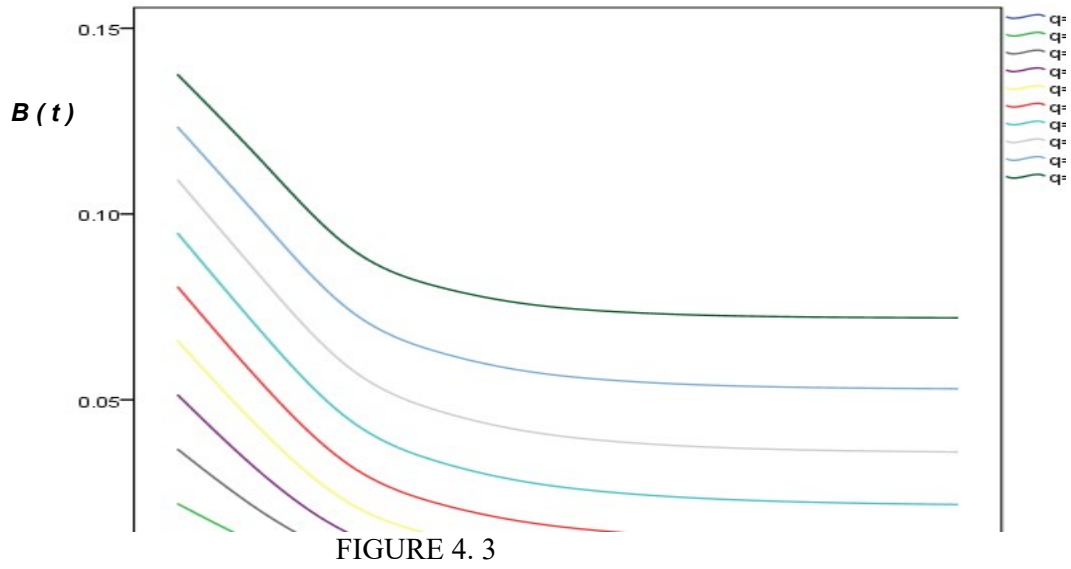


FIGURE 4. 2

TABLE 4. 1 The distribution  $B(t)$  against  $t$  and  $q$

$q \backslash t$	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19
.5	.0073572	.0220630	.0367435	.0513828	.0659661	.0804794	.0949097	.1092447	.1234731	.1375843
1.5	.0031668	.0105784	.0194085	.0296046	.0410842	.0537420	.0674551	.0820902	.0975081	.1135688
2.5	.0008167	.0034481	.0076930	.0138128	.0219570	.0321554	.0443280	.0583061	.0738587	.0907190
3.5	.0001902	.0011294	.0032781	.0072080	.0133894	.0220978	.0333818	.0470863	.0629077	.0804580
4.5	.0000436	.0003983	.0015708	.0042948	.0093103	.0171382	.0279608	.0416320	.0577695	.0758686
5.5	.0000102	.0001538	.0008451	.0028717	.0071755	.0144886	.0251006	.0388530	.0552795	.0737749
6.5	.0000024	.0000651	.0005040	.0021074	.0059659	.0129830	.0235214	.0373916	.0540477	.0728079
7.5	.0000006	.0000301	.0003284	.0016636	.0052383	.0120899	.0226239	.0366085	.0534312	.0723583
8.5	.0000002	.0000151	.0002306	.0013894	.0047811	.0115448	.0221045	.0361844	.0531209	.0721486
9.5	.0000000	.0000081	.0001723	.0012117	.0044847	.0112057	.0218006	.0359534	.0529641	.0720507

Solution of the distribution function for busy period of this model may determine more readily by plotting  $B(t)$  against  $t$  and  $q$  as given in FIGURES 4.3 and 4.4 respectively.



In figures 4.1 and 4.3 the probability density function  $b(t)$  and distribution function  $B(t)$  for busy period goes to zero as time approaches to infinity.

In figures 4.2 and 4.4 the probability density function  $b(t)$  and distribution function  $B(t)$  for busy increase with feedback increasing.

## 5. CONCLUSION

This paper investigated how feedback, Lagrange's equation and probability generating function approach affect on  $M/M/\infty$  queue. Probability density function  $b(t)$  and distribution function  $B(t)$  of busy period were assumed to depend on  $p_1(t)$  and  $p_0(t)$  respectively. A probability generating function approach was devised to determine the probability density function and distribution function of busy period. Busy period self-service system without feedback concept was derived and discussed as special case of this model. Finally, some numerical values of this system were confirmed.

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