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MATHEMATICS

A new Method to Estimate the Parameters of Quadratic Regression

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KEY WORDS

Quadratic Regression, Kuhn-Tucker conditions, Auto-correlation.

ABSTRACT

In this study, a new method to estimate the parameters for a quadratic regression model is introduced by using Kuhn-Tucker conditions. Kuhn-Tucker conditions provide the minimizing error of the estimated parameters for a quadratic regression. This method can be used for any data set of a quadratic regression, and we discuss the test for correct specification of disturbances mainly because of their ability to detect the irregularities in the regressor specification.

1. Introduction

A quadratic regression refers to linear regression with two or more predictors $(x_1, x_2, \ldots, x_n)$. When multiple predictors are used, the regression line cannot be visualized in two-dimensional space. However, the line can be computed simply by expanding the equation for single-predictor linear regression to include the parameters for each of the predictors.

Although linearity is still the primary model in applications, there has been an increasing in examples of data that are nonlinear, and in particular, where a quadratic fit may be more appropriate. The method of Theil [9] can readily be modified for the quadratic case.

Necessary and sufficient conditions for noninferiority due to Kuhn and Tucker are analogous to the classic Kuhn-Tucker conditions for optimality of a scalar optimization problem. The Kuhn-Tucker conditions for noninferiority (KTCN) will be

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defined in the same spirit as in Cohon and Marks [3] and Cohon [2].

A quadratic regression models play an important role in many fields. The object of this paper is to estimate the parameters for a quadratic regression model by using Kuhn-Tucker conditions. Kuhn-Tucker conditions provide the minimize error of the estimated parameters for a quadratic regression.

The Durbin–Watson statistic is a statistical test used to detect the presence of autocorrelation in the residuals (prediction errors) from a regression analysis [9, 10]. Durbin and Watson (1950, 1951) applied this statistic to the residuals from the regression line, and developed bounds tests for the null hypothesis that the errors are serially uncorrelated against the alternative that they follow a first order autoregressive process.

A quadratic regression models which studied by using Kuhn-Tucker conditions can take the following form:

$$y_i = \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i \qquad ,i = 1,2,\ldots,n \qquad (1.1)$$

where y_i, x_i and (β_1, β_2) are vectors of endogenous variables, exogenous variables, and regression parameters to be estimated, respectively, and ε_i is a random error, assumed to be normally distributed, independently of the errors for other observations, with expectation 0 and variance σ^2: $\varepsilon_i \approx N(0,\sigma^2)$.

The purpose of this article is to develop a new procedure that can always produce regression curve estimators for the quadratic model (1.1) by using the Kuhn-Tucker conditions.

2. Problem Formulation

Equation (1.1) can be written in the following form:

$$\begin{aligned} &\min Q = \sum_{i=1}^{n} \varepsilon_i^2 \qquad ,i = 1,\ldots,n \\ &s.t. \\ &\sum_{i=1}^{n} [y_i - (\beta_1 x_i + \beta_2 x_i^2]^2 \le Q, \\ &x_i \ge 0 \end{aligned} \qquad (2.1)$$

Definition1 [1]:

Let $f_i : R^n \to R,\ g_i : R^n \to R$ and $S = \{x \in R^n : g_i \le 0\}$, a feasible solution $x \in S$ is said to be satisfy KTCN for vector optimization problem if:

1. all f_i and g_i are differentiable and $S = \phi$; and
2. there exists $u_i \ge 0,\ i = 1,\ldots,n$, with strict inequality holding for at least one and $v_i \ge 0,\ i = 1,\ldots,n$, such that

   $g_i(x) \le 0, \quad v_i\, g_i(x) = 0 \quad (i = 1,\ldots,n),$

   and

$$\sum_{i=1}^{n} u_i \ \nabla f_i(x) + \sum_{i=1}^{n} v_i \ \nabla g_i(x) = 0.$$

The Kukn-Tucker conditions for this problem take the form (see [6, 7]):

$$u_i \left[-2\sum_{i=1}^{n} [y_i - (\hat\beta_1 x_i + \hat\beta_2 x_i^2)](\hat\beta_1 + 2\hat\beta_2 x_i) \right] - \sum_{i=1}^{n} v_i x_i = 0, \qquad (2.2)$$

$$\sum_{i=1}^{n} u_i = 1 \qquad (2.3)$$

$$\sum_{i=1}^{n} [y_i - (\hat\beta_1 x_i + \hat\beta_2 x_i^2)]^2 \le 0, \qquad (2.4)$$

$$-\sum_{i=1}^{n} v_i x_i = 0, \qquad (2.5)$$

$$x_i \ge 0, \qquad (2.6)$$

$$u_i\left[\sum_{i=1}^{n}[y_i-(\hat{\beta}_1x_i+\hat{\beta}_2x_i^2)]^2\right]=0, \tag{2.7}$$

$$u_i \geq 0, \quad i=1,...,n \tag{2.8}$$

$$v_i \geq 0 \tag{2.9}$$

The determination of $\hat{\beta}_1,\hat{\beta}_2$ is depending on the obtained values of u_i which can be determined as follow:

If $u_i>0, v_i=0$, we have:

$$\hat{\beta}_1=\frac{\overline{y}}{\overline{x}}-\frac{\hat{\beta}_2}{n\overline{x}}\sum_{i=1}^{n}x_i^{\,2}, \qquad \hat{\beta}_2=\frac{\sum_{i=1}^{n}x_iy_i-\frac{\overline{y}}{\overline{x}}\sum_{i=1}^{n}x_i^{\,2}}{\sum_{i=1}^{n}x_i^{\,3}-\frac{\left(\sum_{i=1}^{n}x_i^{\,2}\right)^2}{n\overline{x}}}.$$

3. Stability set of this problem

Given certain $(\hat{\beta}_1,\hat{\beta}_2)$ with corresponding optimal solution e_i ; then the stability set of the first kind of problem (2.1) corresponding to this optimal solution, denoted by $S(e_i)$, is defined by $S(e_i)=\{(\hat{\beta}_1,\hat{\beta}_2)\in R\,|\,\mathrm{e}_i$ is an optimal solution of (2.1)$\}$ (see [6, 7]).

For a certain $(\hat{\beta}_1,\hat{\beta}_2)$ with a corresponding to optimal solution e_i ; we have from the stability of (2.1) there exist $(\hat{\beta}_1,\hat{\beta}_2)\in R,\ u_i \geq 0, i=1,...,n$, such that the Kuhn-Tucker conditions of problem (2.1) takes the form (2.2)-(2.9), the determine of the stability set of the first kind $S(e_i)$ depends only on whether any of the variables u_i, $i=1,...,n$ and any of the variables v,w which solves the Equations (2.2), (2.3), (2.8), (2.9) are positive or zero.

Let $u_i=0, i\in I\subset\{1,...,n\}, u_i>0,\ \ i\notin I$ solves (2.1), (2.2) and (2.7), then in order to satisfy the other Kuhn-Tucker conditions (2.3) and (2.6), we must have

$$e_i=y_i-(\hat{\beta}_1x_i+\hat{\beta}_2x_i^2),\ i\notin I;$$
$$e_i\geq y_i-(\hat{\beta}_1x_i+\hat{\beta}_2x_i^2),\ i\in I.$$

Let

$D=\{I\,|\,\mathrm{u}_i=0, i\in I;\mathrm{u}_i>0,\ \ i\notin I$ solve (2.1), (2.2), (2.7) and (2.8)$\}$

and

$$S_I(e_i)=\{(\hat{\beta}_1,\hat{\beta}_2)\in R\,|\,e_i=y_i-(\hat{\beta}_1x_i+\hat{\beta}_2x_i^2),\ i\notin I;\ e_i\geq y_i-(\hat{\beta}_1x_i+\hat{\beta}_2x_i^2),\ i\in I\}.$$

Then, it is clear that

$$S(e_i)=\underset{I\in D}{}S_I(e_i)$$

4. Testing Disturbances

In this study we examine whether the disturbances in the regression model (2.1) are well behaved or not. As known, this can also be viewed as a test for the higher moments of the dependent variable conform to the assumptions of the model.

Testing for autocorrelation may be the most intensely researched statistical problems in all of econometrics. Despite heavy competitions, the most common procedure is still the Durbin–Watson test. It is based on the assumption that the e_i in (2.1) follows a stationary autoregressive process.

The null hypothesis of no serial correlation is therefore equivalent to $H_0:\rho=0$. The statistic test in the multivariate case as the Durbin–Watson test is

$$D = \frac{\sum_{i=2}^{n} \left(e_i - e_{i-1}\right)^2}{\sum_{i=1}^{n} e_i^2} \qquad (4.1)$$

where $e_i = y_i - \hat{y}_i$ is residuals , y_i and are, respectively, the observed and predicted values of the response variable for individual i. A major problem with the Durbin–Watson test used to be that the rejection region depends not only on the significance level α of the test, but also on the regression vector X. Durbin *et. al.* [5] gave the familiar bounds D_U and for which depends only on α, n such that (when testing against positive serial correlation)

If $D < D_L$ reject $H_0: \rho = 0$

If $D > D_U$ do not reject $H_0: \rho = 0$

If $D_L < D < D_U$ test is inconclusive.

Example

In this example we examined whether the disturbances in the regression median model (2.1) are well behaved or not. According to Durbin–Watson test, therefore the null hypothesis of no serial correlation ($H_0: \rho = 0$), against ($H_1: \rho > 0$).

The data set of X and Y is given in table 1

Table 1:

i	x_i	y_i	$\hat{y}_i(\mathrm{KT}) = 4.235 * x_i + 0.25 * x_i^2$	$e_i = y_i - \hat{y}_i$	e_i^2	e
1	3.4	35	17.289	17.711	313.6795	
2	5.7	40	32.262	7.738	59.87664	[illegible]
3	7.5	42.5	45.825	-3.325	11.05563	[illegible]
4	8.8	44	56.628	-12.628	159.4664	[illegible]
5	11.1	60.5	77.811	-17.311	299.6707	[illegible]
6	12.5	80	92	-12	144	[illegible]
7	14.2	120	110.547	9.453	89.35921	[illegible]
8	15.2	125	122.132	2.868	8.225424	[illegible]
9	15.8	132	129.323	2.677	7.166329	[illegible]
10	17.9	140	155.909	-15.909	253.0963	[illegible]
11	18	145	157.23	-12.23	149.5729	[illegible]
12	18.2	160	159.887	0.113	0.012769	[illegible]
13	19	174	170.715	3.285	10.79123	[illegible]

By using equation (4.1) the statistic test is $D(\mathrm{KT}) = 0.77$, at significances level $\alpha = 0.05$, then table 2 gives the critical values corresponding to and one regressor as $D_L = 1.08$ and .

Since , then we reject $H_0: \rho = 0$ and conclude that the errors are positively autocorrelated.

Table 2: critical values of the Durbin-Watson statistic

Sample size (n)	(Significance Level= α)	k=1		k=2		k=3		k=4
		DL	DU	DL	DU	DL	DU	DL
15	0.01	0.81	1.07	0.7	1.25	0.59	1.46	0.49
	0.025	0.95	1.23	0.83	1.4	0.71	1.61	0.59
	0.05	1.08	1.36	0.95	1.54	0.82	1.75	0.69
20	0.01	0.95	1.15	0.86	1.27	0.77	1.41	0.63
	0.025	1.08	1.28	0.99	1.41	0.89	1.55	0.79
	0.05	1.2	1.41	1.1	1.54	1	1.68	0.9
25	0.01	1.05	1.21	0.98	1.3	0.9	1.41	0.83
	0.025	1.13	1.34	1.1	1.43	1.02	1.54	0.94
	0.05	1.29	1.45	1.21	1.55	1.12	1.66	1.04
30	0.01	1.13	1.26	1.07	1.34	1.01	1.42	0.94
	0.025	1.25	1.38	1.18	1.46	1.12	1.54	1.05
	0.05	1.35	1.49	1.28	1.57	1.21	1.65	1.14
40	0.01	1.25	1.34	1.2	1.4	1.15	1.46	1.1
	0.025	1.35	1.45	1.3	1.51	1.25	1.57	1.2
	0.05	1.44	1.54	1.39	1.6	1.34	1.66	1.29
50	0.01	1.32	1.4	1.28	1.45	1.24	1.49	1.2
	0.025	1.42	1.5	1.38	1.54	1.34	1.59	1.3
	0.05	1.5	1.59	1.46	1.63	1.42	1.67	1.38
60	0.01	1.38	1.45	1.35	1.48	1.32	1.52	1.28
	0.025	1.47	1.54	1.44	1.57	1.4	1.61	1.37
	0.05	1.55	1.62	1.51	1.65	1.48	1.69	1.44
	0.01	1.47	1.52	1.44	1.54	1.42	1.57	1.39

K= number of regressors.

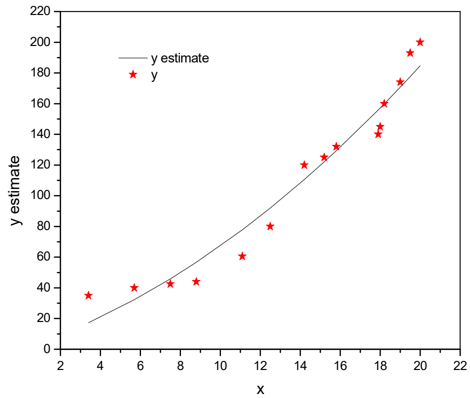

Fig. (1) estimated quadratic regression by Kuhn-Tucker conditions

5. Comparison of Kuhn-Tucker Estimation and Least Squares Estimation on Quadratic Regression

The data in table 1 has been studied by using least square method.

By using SPSS program for this example, we get the following analysis data:

$$\hat{\beta}_1 = 3.062, \qquad \hat{\beta}_2 = 0.320.$$

Then the quadratic regression equation estimated by least square (LS) method given by:

$$\hat{y}_i(\text{LS}) = (3.062)x_i + (0.320)x_i^2 \quad ,i = 1,\ldots,15 \quad (5.1)$$

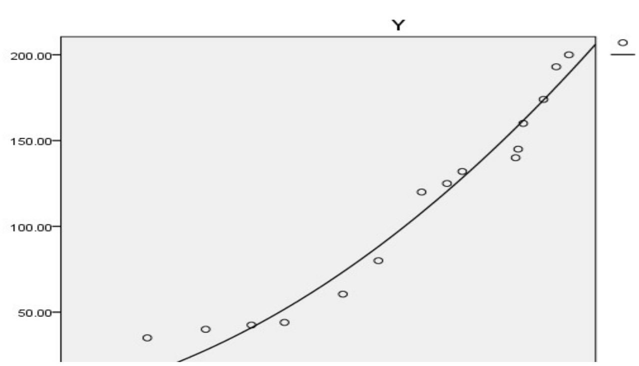

Fig. (2): estimated quadratic regression by least square method

The estimated quadratic regression is illustrated in Fig. (3) as a comparison between Kuhn-Tucker (KT) conditions and least square (LS) method

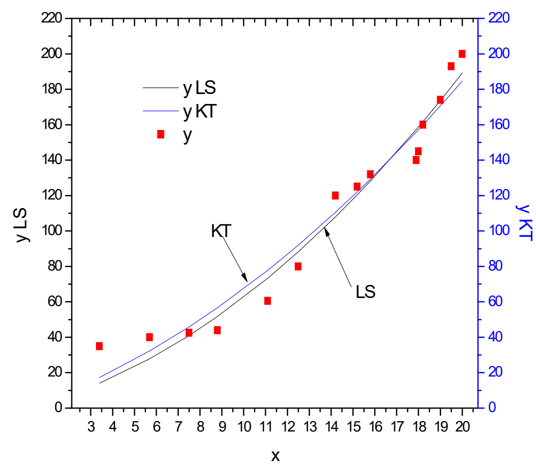

Fig. (3): estimated quadratic regression by Kuhn-Tucker (KT) conditions and least square (LS) method

Residual values can be calculated as shown in the following table (3).

Table (3):

i	y_i	$\hat{y}_i$(LS)	$e_i = y_i - \hat{y}_i$	e_i^2	$e_i - e_{i-1}$
1	35	14.11	20.89	436.3921	
2	40	27.8502	12.1498	147.6176	-8.7402
3	42.5	40.965	1.535	2.356225	-10.6148
4	44	51.7264	-7.7264	59.69726	-9.2614
5	60.5	73.4154	-12.9154	166.8076	-5.189
6	80	88.275	-8.275	68.47563	4.6404
7	120	108.0052	11.9948	143.8752	20.2698
8	125	120.4752	4.5248	20.47382	-7.47
9	132	128.2644	3.7356	13.95471	-0.7892
10	140	157.341	-17.341	300.7103	-21.0766
11	145	158.796	-13.796	190.3296	3.545
12	160	161.7252	-1.7252	2.976315	12.0708
13	174	173.698	0.302	0.091204	2.0272

By using equation (4.1) the statistic test is $D(\text{LS}) = 0.85$, at significances level , table 2 gives the critical values corresponding to $n = 15$ and one regressor as and $D_U = 1.36$.

Since , then we reject and conclude that the errors are positively autocorrelated.

The result shows that, the statistic Durbin–Watson test value in Kuhn-Tucker estimation is less than its value in least square estimation.

Conclusion

In this paper, we introduce a new method to estimate the parameter for a quadratic regression model by using Kuhn-Tucker conditions. According to the Durbin–Watson test we show that there is positively autocorrelation between the errors for the regression curve, this means that our estimators are a suitable estimator in the case of fitting data.

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