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Research Article

MATHEMATICS

The Solution of the Fractional Form of Unsteady Axisymmetric Squeezing Fluid Flow with Slip and No-Slip Boundaries by Analytical Techniques

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KEY WORDS

Squeezing flow; Axisymmetric flow; Variational iteration method; Adomian decomposition method; New iterative method; Picard method and Fractional Calculus

ABSTRACT

In this article, two powerful techniques called variational iteration and Adomian decomposition methods are proposed to solve analytically the fractional form of the fourth order nonlinear ordinary differential equation which reduced by similarity transformation from the basic system of nonlinear partial differential equations of motion of an unsteady axisymmetric flow of nonconducting, Newtonian fluid squeezed between two circular plates with slip and no-slip boundaries. The analysis of convergence of the proposed methods is discussed through the absolute residual errors for various Reynolds number and various values of the fractional order. Comparisons between the results obtained by the proposed methods with those obtained by the new iterative and Picard methods are made which confirm that the proposed methods are powerful methods and therefore suitable for solving this kind of problems.

1. Introduction

The importance of fluid flow through channels began since long time ago in various applications in the life specially in agriculture. Recently, squeezing flow has large attention of the researchers and scientists due to its wide range of applications in various fields like chemical, food industries and in bio-mechanics Practical applications of squeezing flows in the mentioned fields are modeling of lubrication systems, compression and injection modeling and polymer processing. These flows are induced by applying vertical velocities or normal stresses by means of a moving boundary, that can be frequently noted in many hydro-dynamical tools and machines.

The first work in squeezing flows was laid down by Stefan [1] in which he developed an ad-hoc asymptotic solution of Newtonian fluids. An explicit solution of the squeeze flow, considering inertial terms, has done by Thorp [2]. However, Gupta and Gupta showed that the solution in [2] fails to satisfy boundary conditions [3]. In [4] Ran et. al. used the homotopy analysis method to obtain an explicit series solution for the squeezing flow between two infinite plates. Also, by this method in [5] Rashidi et. al. established an analytical solution to the unsteady squeezing flow of a second-grade fluid between two circular plates. Moreover, the processes of polymer extrusion are modeled by squeezing flow of viscous fluids in [6]. Leider and Bird in [7] presented theoretical analysis of power-law fluid between parallel disks. Verma [8] and Singh et. al. [9] have established numerical solutions of squeezing flow between parallel plates. Hemeda and Aladdad in [10] used the new iterative and Picard methods for solving the fractional form of unsteady axisymmetric flow of nonconducting, Newtonian fluid squeezed between two circular plates with slip and no-slip boundaries. Qayyum et. al. present, in [11] analysis of unsteady axisymmetric flow of nonconducting, Newtonian fluid squeezed between two circular plates with slip and no-slip boundaries using OHAM, while in [12], the authors modeling and analyzing the unsteady axisymmetric flow of nonconducting, Newtonian fluid squeezed between two circular platers passing through porous medium channel with slip boundary condition using HPM. Sheikholeslami et. al. in [13] used the Adomian decomposition method to solve the problem governing the unsteady flow of a nanofluid squeezed between two parallel plates and in [14] used the HPM for solving the problem governing the heat transfer of a nanofluid flow squeezed between parallel plates. For more studies, you can see [15-18].

The differential equations of fractional order, are the generalized type of the classical differential equations of integer order. Recently, the fractional differential equations have been the focus of many researchers because of their frequent appearance in many applications in viscoelasticity, physics, biology, engineering and fluid mechanics. Therefore, considerable attention has been given to the solutions of differential equations of fractional order, of fluid mechanics and physics interest. Most nonlinear differential equations of fractional order do not have exact analytic solutions, so numerical and approximation techniques, such as Jacobi collocation method [19], Galerkin method [20], He's frequency-amplitude formulation and energy balance methods [21], max-min and Hamiltonian methods [22, 23], variational iteration method [24-28] and Adomian decomposition method [29-33] must be used. The variational iteration and Adomian decomposition methods, which are special techniques of the homotopy analysis method [34], are relatively new techniques to provide analytical approximation to nonlinear problems and they are particularly valuable as tools for researchers, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to nonlinear differential equations without linearization or discretization.

In last years, the application of the proposed methods in extended for fractional differential equations. The goal of this chapter, is to prepare and use these methods for solving the fractional order form of an unsteady axisymmetric flow of nonconducting, Newtonian fluid squeezed between two circular plates with slip and no-slip boundaries because of the importance of this problem and its wide applications in many subjects in fluid mechanic. In addition, the comparisons between the results obtained by the proposed methods with those obtained by some other methods are considered to confirm the power and efficiency of these methods in solving this kind of problems.

2. Formulation of the Problem

In this section, the unsteady axisymmetric flow of incompressible Newtonian fluid with viscosity μ , density ρ and kinematic viscosity v, squeezed between two circular plates having speed $E_w(t)$ is considered with a fractional form. At any time t, it is assumed that the distance between the two circular plates is $2h(t)$. Also, it is assumed that r-axis is the central axis of the channel while z-axis is taken normal to it. Plates move symmetrically with respect to the central axis

$z=0$ while the flow is axisymmetric about $r=0$. The normal and longitudinal velocity components in axial and radial directions are $w_z(r,z,t)$ and $w_r(r,z,t)$, respectively (Figure 1). For more details, see [10-12].

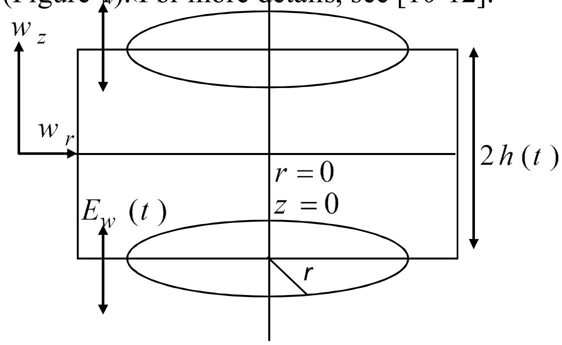

Fig. 1: Geometrical of the problem.

The equations of motion are:

$$\frac{\partial w_r}{\partial r}+\frac{w_r}{r}+\frac{\partial w_z}{\partial z}=0, \qquad (1)$$

$$\frac{\partial p}{\partial r}+\rho\left(\frac{\partial w_r}{\partial t}-w_z\,\Omega\right)=-\mu\frac{\partial \Omega}{\partial z}, \qquad (2)$$

$$\frac{\partial p}{\partial z}+\rho\left(\frac{\partial w_r}{\partial t}+w_r\,\Omega\right)=\frac{\mu}{r}\frac{\partial}{\partial r}(r\,\Omega), \qquad (3)$$

where $\Omega=\Omega(r,z,t)$ is the vorticity function; and $p=p(r,z,t)$ is the generalized pressure.

The boundary conditions on $w_r(r,z,t)$ and $w_z(r,z,t)$ are:

$$w_r(r,z,t)=0, \quad w_z(r,z,t)=E_w(t) \text{ at } z=h$$

$$\frac{\partial}{\partial z}w_r(r,z,t)=0,\ w_z(r,z,t)=0 \text{ at } z=0, \qquad (4)$$

where $E_w(t)=\dfrac{d\,h}{d\,t}$ is the velocity of the plates. The boundary conditions in equation (4) are because of no-slip at the upper plate when $z=h$ and symmetry at $z=0$. If we define the dimensionaless parameter:

$$\eta=\frac{z}{h(t)}, \qquad (5)$$

equations (1), (2) and (3) transforms to:

$$\frac{\partial w_r}{\partial r}+\frac{w_r}{r}+\frac{1}{h}\frac{\partial w_z}{\partial \eta}=0, \qquad (6)$$

$$\frac{\partial p}{\partial r}+\rho\left(\frac{\partial w_r}{\partial t}-w_z\,\Omega\right)=-\frac{\mu}{h}\frac{\partial \Omega}{\partial \eta}, \qquad (7)$$

$$\frac{1}{h}\frac{\partial p}{\partial \eta}+\rho\left(\frac{\partial w_r}{\partial t}+w_r\,\Omega\right)=\frac{\mu}{r}\frac{\partial}{\partial r}(r\,\Omega). \qquad (8)$$

The boundary conditions on w_r and w_z are:

$$w_r=0, \qquad w_z=E_w(t) \qquad \text{at } \eta=1$$

$$\frac{\partial}{\partial \eta}w_r=0, \qquad w_z=0 \qquad \text{at } \eta=0. \qquad (9)$$

By eliminating he generalized pressure between equations (7) and (8), we obtain:

$$\rho\left[\frac{\partial \Omega}{\partial t}+w_r\frac{\partial \Omega}{\partial r}+\frac{w_z}{h}\frac{\partial \Omega}{\partial \eta}-\frac{w_r}{r}\Omega\right]=\mu\left[\nabla^2\Omega-\frac{\Omega}{r^2}\right], \qquad (10)$$

where ∇^2 is the Laplacian operator.

Defining velocity components as [3, 5]:

$$w_r=-\frac{r}{2h(t)}E_w(t)u'(\eta), \qquad w_z=E_w(t)u(\eta), \qquad (11)$$

we see that equation (6) is identically satisfied and equation (10) becomes:

$$\frac{d^4u}{d\eta^4}+R\left[(\eta-u)\frac{d^3u}{d\eta^3}+2\frac{d^2u}{d\eta^2}\right]-Q\frac{d^2u}{d\eta^2}=0, \qquad (12)$$

where

$$R=\frac{h\,E_w}{v} \text{ and } Q=\frac{h^2}{v\,E_w}\frac{d\,E_w}{d\,t}. \qquad (13)$$

In equations (12) and (13) R and Q are functions of t but for similarity solution we consider these functions are constants. Since $E_w=\dfrac{d\,h}{d\,t}$. Integrate first equation of equation (13), we get:

$$h(t)=(c\,t+d)^{\frac{1}{2}}, \qquad (14)$$

where c and d are constants. The plates move from each other symmetrically with respect to η when $c>0$ and $d>0$. Also, the plates approach to each other and squeezing flow exists with similar velocity profiles when $c<0$, $d>0$ and $h(t)>0$. From equations (13) and (14) it follows that $Q=-R$. Therefore equation (12) becomes:

$$\frac{d^4u}{d\eta^4}+R\left[(\eta-u)\frac{d^3u}{d\eta^3}+3\frac{d^2u}{d\eta^2}\right]=0. \quad (15)$$

Using equations (9) and (11), we can define the boundary conditions at the upper plate in case of no-slip and slip as follows:

No-slip at the wall:

$$u(1)=1,\quad u'(1)=0,\quad u(0)=0,\quad u''(0)=0 \quad (16a)$$

Slip at the wall:

$$u(1)=1,\quad u'(1)=\gamma u''(1),\ u'(0)=0,\quad u''(0)=0\ . \quad (16b)$$

3. Fractional Calculus

Now we mention basic definitions of fractional calculus which are used in the present work.

Definition 3.1: The fractional integral operator of order $\alpha>0$, of a function $f(t)\in c_\mu$ and $\mu\geq 1$, in the Riemann-Liouville sense is defined as [35]:

$$I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}f(\tau)\,d\tau,\qquad \alpha>0,\ t>0$$

$$I_t^0 f(t)=f(t). \quad (17)$$

For the Riemann-Liouville fractional integral operator, I_t^α, we obtain:

$$I_t^\alpha t^v=\frac{\gamma(v+1)t^{v+\alpha}}{\Gamma(v+1+\alpha)}. \quad (18)$$

Definition 3.2: The fractional derivative of $f(t)$ in the Caputo sense is defined as [36]:

$$D_t^\alpha f(t)=I^{m-\alpha}D^{tm}$$

$$=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-r)^{m-\alpha-1}f^{(m)}(\tau)d\tau \quad (19)$$

For the Caputo fractional derivative operator, D_t^α, we obtain:

$$D_t^\alpha t^v=\frac{\Gamma(v+1)t^{v-\alpha}}{\Gamma(v+1-\alpha)}. \quad (20)$$

For the Riemann-Liouvill fractional integral and Caputo fractional derivative of order α, we have the following relations:

Lemma 1: If $m-1<\alpha\leq m$, $m\in N$ and $f\in C_\mu^m$, $\mu\geq -1$, then:

$$D_t^\alpha I_t^\alpha f(t)=f(t)\ , \quad (21a)$$

$$D_t^\alpha I_t^\alpha f(t)=\begin{cases} f(t)-\sum_{k=0}^{m-1} f^{(k)}(0_+)\frac{t^k}{k!}, & t>0,\ m-1<\deg\cdot f(t)\\ 0, & m-1\geq \deg\cdot f(t)\end{cases} \quad (21b)$$

Remark 1: According to the previous fractional calculus, equation (15) can be rewrite in the following fractional order form:

$$\frac{d^\alpha u}{d\eta^\alpha}+R\left[(\eta-u)\frac{d^3u}{d\eta^3}+3\frac{d^2u}{d\eta^2}\right]=0,\ 3<\alpha\leq 4. \quad (22)$$

4. Analysis of the Considered Methods

In this section, we discuss the considered methods with preparing them for solving any fractional differential equation.

4.1. Variational Iteration Methods (VIM):

To illustrate the basic idea of this method, let us consider the following general differential equation of fractional order [24-28]:

$$D_t^\alpha u(t)=R[t]u(t)+g(t),\quad m-1<\alpha\leq m,\ m\in N \quad (23a)$$

subject to the initial values:

$$\frac{d^k}{dt^k}u(0)=h_k,\qquad k=0,1,2,\ldots,m-1, \quad (23b)$$

where $R[t]$ is a nonlinear differential operator in t, $g(t)$, $h_k(t)$ are given continuous functions and D_t^α is the fractional differential operator of order $\alpha>0$ in the

Caputo sense. According to the VIM, we can construct the correction functional for equation (23) as:

$$u_{n+1}(t)=u_n(t)+I_t^{\beta}[\lambda[D_t^{\alpha}u_n(t)-R[t]\tilde{u}_n(t)-g(t))],$$
$$=u_n(t)+\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}\lambda(\tau)[D_t^{\alpha}u_n(\tau)-R[\tau]\tilde{u}_n(\tau)$$
$$-g(\tau))]\,d\tau, \qquad (24)$$

where I_t^{β} is the fractional integral operator o Riemann-Liovill of order $\beta=\alpha+1-m$, which respect to t and λ is a general Lagrange multiplier, which can be identified optimally via variational theory, to identify λ approximately, we must made some approximation. Therefore, the correction functional (24) can be approximately in the form:

$$u_{n+1}(t)=u_n(t)+\int_0^t[\lambda(\tau)(D_t^{\alpha}u_n(\tau)-R[\tau]\tilde{u}_n(\tau)-g(\tau))]\,d\tau \qquad (25)$$

Here we apply restricted variations to the nonlinear term $R[t]u$, in this case we can easily determine the multiplier. Making the above functional stationary, noticing that $\delta\tilde{u}_n=0$,

$$\delta u_{n+1}(t)=\delta u_n(t)+\delta\int_0^t\lambda(\tau)[D_t^{\alpha}u_n(\tau)-g(\tau)]\,d\tau, \qquad (26)$$

yield the following Lagrange multiplier:

$$\begin{cases}\lambda(\tau)=-1, & \text{for} \quad m=1,\\ \lambda(\tau)=\tau-t, & \text{for} \quad m=2,\\ \lambda(\tau)=-\frac{1}{2}(\tau-t)^2, & \text{for} \quad m=3,\\ \lambda(\tau)=\frac{1}{6}(\tau-t)^3, & \text{for} \quad m=4.\end{cases} \qquad (27)$$

Generally, and in closed form, equation (27) becomes:

$$\lambda(\tau)=\frac{(-1)^m}{\Gamma(m)}(\tau-t)^{m-1}, \qquad m\geq 1. \qquad (28)$$

Therefore, for $m=1\ (0<\alpha\leq 1)$, we substitute $\lambda=-1$ into (24) to obtain the iterative formula:

$$u_{n+1}(t)=u_n(t)-I_t^{\alpha}[D_t^{\alpha}u_n(t)-R[\tau]u_n(t)-g(t)],\ n\geq 0 \qquad (29)$$

For $m=2\ (1<\alpha\leq 2)$, we substitute $\lambda=\tau-t$ into (24) to obtain the iterative formula:

$$u_{n+1}(t)=u_n(t)-(\alpha-1)I_t^{\alpha}[D_t^{\alpha}u_n(t)-R[\tau]u_n(t)-g(t)],\ n\geq 0 \qquad (30)$$

For $m=3\ (2<\alpha\leq 3)$, we substitute $\lambda=-\frac{1}{2}(\tau-t)^2$ into (24) to obtain the iterative formula:

$$u_{n+1}(t)=u_n(t)-\frac{1}{2}(\alpha^2-3\alpha+2)I_t^{\alpha}[D_t^{\alpha}u_n(t)-R[\tau]u_n(t)-g(t)],\ n\geq 0 \qquad (31)$$

Finally, for $m=4\ (3<\alpha\leq 4)$, we substitute $\lambda=\frac{1}{6}(\tau-t)^3$ into (24) to obtain the iterative formula:

$$u_{n+1}(t)=u_n(t)-\frac{1}{6}(\alpha^3-6\alpha^2+11\alpha-6)I_t^{\alpha}[D_t^{\alpha}u_n(t)-R[\tau]u_n(t)$$
$$-g(t)],\ n\geq 0. \qquad (32)$$

The zero solution u_0 can be freely chosen if it satisfies most of the given conditions. However, the success of the method depends on the proper selection of the zero solution u_0. Finally, we approximate the solution $u(t)=\lim_{n\to\infty}u_n(t)$ by the n^{th}-*term* $u_n(t)$.

4.2. Adomian Decomposition Method (ADM):

To illustrate the basic idea of this method, let us consider the fractional differential equation [29-33]:

$$D_t^{\alpha}u(t)=Lu(t)+Nu(t)+g(t),\quad m-1<\alpha\leq m,\ m\in N \qquad (33a)$$

with the initial values:

$$\frac{d^k}{dt}u(0)=h_k, \qquad k=0,1,2,\dots,m-1, \qquad (33b)$$

where L is linear operator and N is nonlinear operator. The method is based on applying the fractional integral operator I_t^{α}, the inverse of the fractional differential operator D_t^{α}, to both sides of (33) to obtain:

$$u(t)=\sum_{k=0}^{m-1}h_k\cdot\frac{t^k}{k!}+I_t^{\alpha}[Lu(t)+Nu(t)+g(t)]. \qquad (34)$$

The ADM suggests the solution $u(t)$ be decomposed into the infinite series of components:

$$u(t)=\sum_{n=0}^{\infty} u_n(t), \qquad (35)$$

and the nonlinear term $N\,u(t)$ in equation (33a) is decomposed as:

$$N\,u=\sum_{n=0}^{\infty} A_n, \qquad (36)$$

where A_n are the so-called Adomian polynomials. Substituting the decomposition series (35) and (36) into both side of (34) gives:

$$\sum_{n=0}^{\infty} u_n(t)=\sum_{k=0}^{m-1} h_k\cdot\frac{t^k}{k!}+I_t^{\alpha}\left[L\sum_{n=0}^{\infty}u_n(t)+\sum_{n=0}^{\infty}A_n+g(t)\right]. \qquad (37)$$

According to the ADM, we can introduce the recurrence relation as:

$$u_0(t)=\sum_{k=0}^{m-1} h_k\cdot\frac{t^k}{k!}+I_t^{\alpha}[g(t)],$$

$$u_{j+1}(t)=I_t^{\alpha}[Lu_j(t)+A_j], \qquad j\ge 0, \qquad (38)$$

The polynomial A_n of Adomian can be calculated for all nonlinear terms via specific constructed by Adomian. Therefore, the general form of A_n is:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{n}\lambda^k u_k\right)\right]_{\lambda=0}. \qquad (39)$$

This formula is easy to compute. Finally, we approximate the solution $u(t)$ by the truncated series:

$$\phi_N(t)=\sum_{j=0}^{N-1} u_j(t) \quad \text{and}$$

$$\lim_{N\to\infty}\phi_N(t)=u(t). \qquad (40)$$

5. Applications:

In the following, we illustrate the application of the two-considered method to solve the nonlinear fractional order ordinary differential equation (22) subject to the boundary conditions (16).

5.1. VIM:

Using Equations (22) and (16), the initial value fractional order problem:

$$\frac{d^{\alpha}}{dt^{\alpha}}u(\eta)+R\left[(\eta-n)\frac{d^3u}{d\eta^3}+3\frac{d^2u}{d\eta^2}\right]=0,\ 3<\alpha\le 4,$$

$$u(0)=0,\quad u'(0)=a,\quad u''(0)=0,\quad u'''(0)=b, \qquad (41)$$

according to the iteration formula (32), becomes:

$$u_{n+1}(h)=u_n(\eta)-\frac{1}{6}(\alpha^3-6\alpha^2+11\alpha-6)I_{\eta}^{\alpha}\left[R\left((\eta-u_n)\frac{d^3u_n}{d\eta^3}+3\frac{d^2u_n}{d\eta^2}\right)\right]. \qquad (42)$$

Beginning with the initial solution $u_0(\eta)=a\eta+\dfrac{b\eta^3}{6}$, which satisfies the given initial conditions, we can obtain the following first few components of the solution:

$$u_1(\eta)=a\eta+\frac{b\eta^3}{6}-\frac{1}{6}(\alpha^3-6\alpha^2+11\alpha-6)\left(\frac{4bR^2\eta^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{abR\eta^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{b^2R\eta^{3+\alpha}}{\Gamma(4+\alpha)}\right),$$

$$u_2(\eta)=u_1+\frac{1}{6^2}(\alpha^3-6\alpha^2+11\alpha-6)^2\left(\frac{4bR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)}-\frac{5abR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)}-\frac{2b^2R^2\Gamma(2+\alpha)\eta^{1+2\alpha}}{3\Gamma(-1+\alpha)\Gamma(2+2\alpha)}\frac{a^2b^2R^2\Gamma(2+\alpha)\eta^{1+2\alpha}}{6\Gamma(-1+\alpha)\Gamma(2+2\alpha)}+\ldots\right),$$

$$u_3(\eta)=a\eta+\frac{b\eta^3}{6}-\frac{1}{6}(\alpha^3-6\alpha^2+11\alpha-6)\left(\frac{4bR\eta^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{abR\eta^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{b^2R\eta^{3+\alpha}}{\Gamma(4+\alpha)}\right)+\frac{1}{6^2}(\alpha^3-6\alpha^2+11\alpha-6)^2\left(\frac{4bR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)}\frac{5abR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)}+\ldots\right), \qquad (43)$$

and so on. In the same way, we can obtain the higher components of the solution.

I the special case, $\alpha = 4$, equation (43) gives:

$$u(\eta) = a\eta + \frac{b\eta^3}{6} - \frac{bR\eta^5}{30} + \frac{abR\eta^5}{120} + \frac{bR\eta^7}{5040} + \frac{bR^2\eta^7}{210} - \frac{abR^2\eta^7}{280} + \frac{a^2bR^2\eta^7}{1680} - \ldots \quad (44)$$

Using the boundary conditions in equation (16) with the initial conditions in equation (41), the unknown's a and b for fixed values of R in equation (44) can be easily determined. In case of no-slip boundary, then $a = 1.5$ and $b = -30$. For $R = 0.5$, the solution is:

$$u(\eta) = 1.5\eta - 0.5\eta^3 + 0.03125\eta^5 + 0.000334821\eta^7 - 0.000171673\eta^9 + 1.02166\times10^{-6}\eta^{11} + 1.36077\times10^{-6}\eta^{13} - 2.6881\times10^{-8}\eta^{15} - 2.68729\times10^{-9}\eta^{17} + 1.23186\times10^{-10}\eta^{19} + 3.42474\times10^{-12}\eta^{21} - 1.79838\times10^{-13}\eta^{23} - 4.69921\times10^{-15}\eta^{25} + 7.12005\times10^{-17}\eta^{27} + 1018728\times10^{-18}\eta^{29} + 1.18008\times10^{-20}\eta^{31}, \quad (45)$$

and in case of slip boundary with $\gamma = 1$, then $a = 0.75$ and $b = 1.5$. For $R = 0.5$, the solution is:

$$u(\eta) = 0.75\eta - 0.25\eta^3 + 0.0203125\eta^5 + 0.00113002\eta^7 - 8.87674\times10^{-5}\eta^9 + 6.753994\times10^{-6}\eta^{11} - 4.50029\times10^{-7}\eta^{13} + 2.54254\times10^{-8}\eta^{15} - 1.00801\times10^{-9}\eta^{17} + 3.48556\times10^{-11}\eta^{19} - 1..05106\times10^{-12}\eta^{21} + 2.53779\times10^{-14}\eta^{23} - 4.22141\times10^{-16}\eta^{25} + 4.23058\times10^{-18}\eta^{27} - 2.22146\times10^{-20}\eta^{29} + 4.6097\times10^{-23}\eta^{31} \quad (46)$$

5.2. ADM:

According to the recurrence relation (38), the initial value fractional order problem (41) gives:

$$u_0(\eta) = a\eta + \frac{b\eta^3}{6},$$
$$u_{j+1}(\eta) = -I_t^{\alpha}[R(t u_j''' + 3u_j'' - A_j)], \qquad j \geq 0, \quad (47)$$

and therefore, the first few components of the solution are as follows:

$$u_0(\eta) = a\eta + \frac{b\eta^3}{6},$$
$$u_1(\eta) = \frac{4bR\eta^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{abR\eta^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{b^2R\eta^{3+\alpha}}{\Gamma(4+\alpha)},$$
$$u_2(\eta) = \frac{4bR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \frac{abR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \frac{b^2R^2(1+\alpha)\eta^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{12bR^2\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \frac{3abR^2\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \ldots$$

etc. In the same way, we can be obtain the higher components of the solution. The 4-term approximate solution is

$$u(\eta) = \sum_{i=0}^{3} u_i(\eta)$$
$$= a\eta + \frac{b\eta^3}{6} + \frac{abR\eta^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{4bR\eta^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{b^2R^2\eta^{3+\alpha}}{\Gamma(2+\alpha)} + \frac{4bR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \frac{abR^2(\alpha-1)\eta^{-1+2\alpha}}{\Gamma(2\alpha)} - \frac{b^2R^2(\alpha+1)\eta^{1+2\alpha}}{\Gamma(2\alpha)} + \ldots \quad (48)$$

In the special case, $\alpha = 4$, equation (48) gives:

$$u(\eta) = a\eta + \frac{b\eta^3}{6} - \frac{bR\eta^5}{30} + \frac{abR\eta^5}{120} + \frac{b^2R\eta^7}{5040} + \frac{bR^2\eta^7}{210} - \frac{abR^2\eta^7}{280} + \frac{a^2bR^2\eta^7}{1680} - \ldots \quad (49)$$

Similarly, using the boundary conditions in equation (16) with the initial conditions in equation (41), the solution for no-slip boundary, at $R = 0.5$, is:

$$u(\eta) = 1.5\eta - 0.5\eta^3 + 0.03125\eta^5 + 0.000334821\eta^7 + 0.000171673\eta^9 + 1.02166\times10^{-6}\eta^{11} + 1.48018\times10^{-6}\eta^{13} + 4.88258\times10^{-8}\eta^{15}, \quad (50)$$

and for slip boundary, at $R = 0.5$, is

$$u(\eta) = 0.75\eta + 0.25\eta^3 - 0.0203125\eta^5 + 0.00113002\eta^7 - 8.87674\times10^{-5}\eta^9 + 6.75394\times10^{-6}\eta^{11} - 2.78821\times10^{-7}\eta^{13} + 3.05161\times10^{-9}\eta^{15}. \quad (51)$$

It is clear that the number of terms of the solution obtained by ADM in equations (48), (49), (50) and (51) are less than the number obtained by VIM in equations (43), (44), (45) and (46).

The residual error of the problem is:

$$\mathrm{Re}(\eta) = \text{Residual Error} = \frac{d^{\alpha}\hat{u}}{d\eta^{\alpha}} + R\left[(\eta-\hat{u})\frac{d^{3}\hat{u}}{d\eta^{3}} + 3\frac{d^{2}\hat{u}}{d\eta^{2}}\right], \quad (52)$$

where $\hat{u}$ is the 4-term approximate solution in equation (43) or equation (48) for equation (41).

If $\mathrm{Re}(\eta)=0$, then the approximate solution $\hat{u}$ becomes the exact solution u. However, in nonlinear problems, this does not occur.

It is clear from the obtained results that the above considered methods are used simply and accurately without linearization or discretisation with their difficulties. Therefore, these methods are powerful methods for solving the nonlinear fractional order differential equations.

5.3. NIM and PM:

The initial value fractional order problem (41) have been solved by Hemeda and Eladdad in [10] using the new iterative method (NIM); introduced by Daftaedar-Gejji and Jafari and Picard method (PM); introduced by Emile Picard, where the two methods are powerful techniques to give analytical approximate solutions for nonlinear problems and they are particularly valuable as tools for researchers, because they give immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to nonlinear differential equations without lineariza-tion or discretisation. The 4-term approximate solutions at $\alpha=4.0$, $R=0.5$, in case of no-slip boundary with $a=1.5$, $b=-3.0$, takes the form:

$$\begin{aligned}u(\eta) &= 1.5\eta - 0.5\eta^{3} + 0.03125\eta^{5} + 0.000334821\eta^{7} - 0.000171673\eta^{9} \\ &+1.02166\times10^{-6}\eta^{11} + 1.36077\times10^{-6}\eta^{13} - 2.6881\times10^{-8}\eta^{15} \\ &-2.68729\times10^{-9}\eta^{17} + 1.23186\times10^{-10}\eta^{19} + 3.42474\times10^{-12}\eta^{21} \\ &-1.79838\times10^{-13}\eta^{23} - 4.69921\times10^{-15}\eta^{25} + 7.12005\times10^{-17}\eta^{27} \\ &+2.18728\times10^{-18}\eta^{29} + 1.18008\times10^{-20}\eta^{31},\end{aligned} \quad (53)$$

and in case of slip boundary with $\gamma=1$, $a=0.75$, $b=1.5$, takes the form:

$$\begin{aligned}u(\eta) &= 0.75\eta + 0.25\eta^{3} - 0.023125\times10^{-6}\eta^{115} + 0.00113002\times10^{-6}\eta^{117} \\ &-8.87674\times10^{-5}\eta^{9} + 6.75394\times10^{-6}\eta^{11} - 4.50029\times10^{-7}\eta^{13} \\ &+2.4254\times10^{-8}\eta^{15} - 1.00801\times10^{-9}\eta^{17} + 3.48554\times10^{-11}\eta^{19} \\ &-1.05106\times10^{-12}\eta^{21} + 2.53779\times10^{-14}\eta^{23} - 4.22141\times10^{-16}\eta^{25} \\ &+4.23058\times10^{-18}\eta^{27} - 2.22146\times10^{-23}\eta^{29} + 4.6097\times10^{-23}\eta^{31}\end{aligned} \quad (54)$$

It is clear that the solutions obtained by VIM in equations (45) and (46) are the same solutions as obtained y NIM and PM in equations (53) and (54) respectively which means that the three methods are identical in solving this problem while, the solution obtained by ADM in equations (50) and (51) is with simple formulation than these methods. Also, it is clear that the solutions obtained by VIM and NIM and PM are polynomials of degree 31, while the solution obtained by ADM is a polynomial of degree 15 which means that the ADM is simpler in its procedures and computations than these methods.

6. Numerical Results and Discussion:

In this chapter, the unsteady axisymmetric flow of nonconducting, incompressible Newtonian fluid squeezed between two circular plates is considered. The resulting nonlinear fractional order boundary value problem are solved analytically in cases of no-slip and slip boundaries using the VIM and ADM.

Tables 1, 2, and 3 represent the approximate solutions in case of no-slip boundary for different values of the fractional order α at fixed value of the Reynolds number R with the corresponding absolute residual errors $|\mathrm{Re}|$ at $\alpha=4.0$ in each table, while Tables 4, 5, and 6 are in case of slip boundary at the slip parameter $\gamma=1$. It is clear from the results in the size tables that as $\alpha\to4$, $u_{\mathrm{VIM}}\to u_{\mathrm{ADM}}$. Therefore, the approximate solutions converge to the exact solutions for this problem. Also, the six tables indicate that the $|\operatorname{Re}|$ for VIM, in case of no-slip boundary, are less than that for ADM, while the inverse in case of slip boundary.

Moreover, Tables 7, 8, and 9, reflect the comparisons between the $|\operatorname{Re}|$ of the VIM,

methods are identical in solving this problem, while the $\gamma = 1.0$ are less for ADM in case of slip boundary than those for the other three methods and more in case of no-slip boundary.

Figure 2a shows the residual errors, Re, for various Reynolds number $R = 0.2,\ 0.4,\ 0.6$ at $\alpha = 0.4$, in case of no-slip boundary, obtained by VIM, while Figure 2b, in case of slip boundary, obtained by ADM with $\gamma = 0.9$. In addition to the above Figures, the effect of Reynolds number R on velocity profiles by VIM in case of no-slip boundary is depicted in Figures 3. In these profiles, R is varied as $R = 0.2,\ 0.4,\ 0.6$ and noted that the normal velocity is increased with increasing R (Figure 3a). Also, it is observed that the

ADM with those for the NIM and PM [9] for various values of $R = 0.1,\ 0.3,\ 0.5$, in case of no-slip and slip boundaries at $\alpha = 4.0$, $\gamma = 1.0$. It is clear from the results in the three tables that the $\gamma = 1.0$ are equal for VIM, NIM and PM which means that the three normal velocity monotonically increases from $\eta = 0$ to $\eta = 1$ for fixed R at a given time. Figure 3b illustrates the effect of R on the longitudinal velocity. It is clear that this component of velocity decreases near the wall but increases near the central axis of the channel.

The effect of Reynolds number R on velocity profiles by ADM in case of slip boundary is shown in Figures 4 for $R = 0.2,\ 0.4,\ 0.6$ at fixed slip parameter $\gamma = 0.9$. It is observed that the normal velocity decreases as R increases (Figure 4a). Also, it is noted that the longitudinal velocity decreases near the central axis of the channel but increases near the walls when R increases (Figure 4b).

Table 1: Solution for different values of η and α at R = 0.2, and the corresponding absolute residual errors at α = 4.0 in case of **no-slip** boundary by the two methods.

η	VIM			ADM		
	α = 3.9	α = 4.0	$\|Re\|$	$\|Re\|$	α = 4.0	α = 3.9
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.149500	0.149500	9.00113^{-14}	3.69149^{-14}	0.149500	0.149500
0.2	0.296005	0.296004	1.15501^{-11}	1.66173^{-11}	0.296004	0.296006
0.3	0.436531	0.436530	1.83062^{-10}	9.63263^{-10}	0.436530	0.436541
0.4	0.568138	0.568128	9.57782^{-10}	1.55055^{-8}	0.568128	0.0568167
0.5	0.687912	0.687893	4.23159^{-10}	1.34707^{-7}	0.687893	0.687998
0.6	0.793009	0.792979	3.88958^{-8}	8.06399^{-7}	0.792979	0.793220
0.7	0.880653	0.880622	3.19774^{-7}	3.75394^{-6}	0.880622	0.881103
0.8	0.948155	0.948148	1.71883^{-6}	1.45616^{-5}	0.948148	0.949021
0.9	0.992923	0.992998	7.29223^{-6}	4.91485^{-5}	0.992998	0.994466
1.0	1.012480	1.012740	2.62701^{-5}	1.48578^{-4}	1.012740	1.015070

Table 2: Solution for different values of η and α at R = 0.4, and the corresponding absolute residual errors at α = 4.0 in case of **no-slip** boundary by the two methods.

η	VIM			ADM		
	α = 3.9	α = 4.0	$\|Re\|$	$\|Re\|$	α = 4.0	α = 3.9
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.149500	0.149500	1.44001^{-12}	5.90750^{-13}	0.149500	0.149500
0.2	0.296009	0.296008	1.84792^{-10}	2.65690^{-10}	0.296008	0.296011
0.3	0.436567	0.436561	2.92946^{-9}	1.53937^{-8}	0.436561	0.4365811
0.4	0.568275	0.568257	1.53937^{-8}	2.47570^{-7}	0.568257	0.568333

0.5	0.688322	0.688284	5.26460^{-9}	2.14804^{-6}	0.688284	0.688494
0.6	0.794012	0.793953	6.02680^{-7}	1.28364^{-5}	0.793953	0.794431
0.7	0.882788	0.882727	4.94654^{-6}	5.96207^{-5}	0.882727	0.883678
0.8	0.952261	0.952252	2.64012^{-5}	2.30619^{-4}	0.952252	0.953969
0.9	1.000230	1.000390	1.10933^{-4}	7.75726^{-4}	1.000390	1.003260
1.0	1.024720	1.025250	3.94994^{-4}	2.33556^{-3}	1.025250	1.029780

Table 3: Solution for different values of η and α at R = 0.6, and the corresponding absolute residual errors at α = 4.0 in case of **no-slip** boundary by the two methods.

η	**VIM**			**ADM**		
	α **= 3.9**	α **= 4.0**	$\lvert Re \rvert$	$\lvert Re \rvert$	α **= 4.0**	α **= 3.9**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.149500	0.149500	7.29050^{-12}	2.99094^{-12}	0.149500	0.149501
0.2	0.296014	0.296012	9.35462^{-10}	1.34411^{-9}	0.296012	0.296017
0.3	0.436601	0.436591	1.48328^{-8}	7.78368^{-8}	0.436591	0.436622
0.4	0.568412	0.568384	7.81776^{-8}	1.25071^{-6}	0.568384	0.568498
0.5	0.688731	0.688673	1.91467^{-8}	1.08378^{-5}	0.688673	0.0688987
0.6	0.795008	0.794921	2.95360^{-6}	6.6451^{-5}	0.794921	0.795632
0.7	0.884903	0.884815	2.42049^{-4}	2.99604^{-4}	0.884815	0.886556
0.8	0.956317	0.956311	1.28279^{-4}	1.15562^{-3}	0.956311	0.958846
0.9	1.007430	1.007680	5.33793^{-4}	3.87382^{-3}	1.007680	1.011900
1.0	1.036710	1.033753	1.87837^{-3}	9.16160^{-3}	1.033753	1.044130

Table 4: Solution for different values of η and α at R = 0.2, and the corresponding absolute residual errors at α = 4.0, γ = 1.0 in case of **slip** boundary by the two methods.

η	**VIM**			**ADM**		
	α **= 3.9**	α **= 4.0**	$\lvert Re \rvert$	$\lvert Re \rvert$	α **= 4.0**	α **= 3.9**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.075250	0.075250	1.19397^{-11}	1.15647^{-11}	0.075250	0.075250
0.2	0.151997	0.151997	1.46091^{-9}	1.42272^{-9}	0.151997	0.151996
0.3	0.231728	0.231730	2.41523^{-8}	2.26996^{-8}	0.231730	0.231724
0.4	0.315911	0.315917	1.72712^{-7}	1.53650^{-7}	0.315917	0.318593
0.5	0.405986	0.405998	7.75157^{-6}	6.35861^{-7}	0.405998	0.405930
0.6	0.503356	0.503375	2.57633^{-6}	1.87467^{-6}	0.503375	0.503222
0.7	0.609384	0.609403	6.92273^{-6}	4.19340^{-6}	0.609403	0.609101
0.8	0.725384	0.725385	1.58395^{-5}	7.06699^{-6}	0.725385	0.724843
0.9	0.852613	0.852560	3.18898^{-5}	7.55604^{-6}	0.852560	0.851657
1.0	0.992270	0.992098	5.77312^{-5}	2.26285^{-6}	0.992098	0.990682

Table 5: Solution for different values of η and α at R = 0.4, and the corresponding absolute residual errors at α = 4.0, γ = 1.0 in case of **slip** boundary by the two methods.

η	**VIM**			**ADM**		
	α **= 3.9**	α **= 4.0**	$\lvert Re \rvert$	$\lvert Re \rvert$	α **= 4.0**	α **= 3.9**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.075250	0.075250	1.86236^{-10}	1.85035^{-10}	0.075250	0.075250
0.2	0.151994	0.151995	2.33752^{-8}	2.27643^{-8}	0.151995	0.151993
0.3	0.231707	0.231711	3.86500^{-7}	3.63261^{-7}	0.231711	0.231698
0.4	0.315823	0.315835	2.76485^{-6}	2.45995^{-6}	0.315835	0.315786

0.5	0.405724	0.4057	1.24188^{-5}	1.01911^{-5}	0.405748	0.405614
0.6	0.502721	0.502757	4.13358^{-5}	3.01175^{-5}	0.502757	0.502457
0.7	0.608044	0.608079	1.11349^{-4}	6.77268^{-5}	0.608079	0.607488
0.8	0.722830	0.722827	2.55786^{-4}	1.15643^{-4}	0.722827	0.721775
0.9	0.848112	0.847998	5.18105^{-4}	1.29590^{-4}	0.847998	0.846262
1.0	0.984815	0.984459	9.46358^{-5}	1.08810^{-5}	0.984459	0.981766

Table 6: Solution for different values of η and α at R = 0.6, and the corresponding absolute residual errors at α = 4.0, γ = 1.0 in case of **slip** boundary by the two methods.

η	**VIM**			**ADM**		
	$\alpha = 3.9$	$\alpha = 4.0$	$\lvert Re \rvert$	$\lvert Re \rvert$	$\alpha = 4.0$	$\alpha = 3.9$
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	0.075250	0.075250	9.42822^{-10}	9.36742^{-12}	0.075250	0.075250
0.2	0.151989	0.151992	1.18341^{-7}	1.15248^{-7}	0.151992	0.151989
0.3	0.231671	0.231691	1.95698^{-6}	1.8335^{-6}	0.231691	0.231671
0.4	0.315680	0.315753	1.40044^{-5}	1.24613^{-5}	0.315753	0.315680
0.5	0.405302	0.405500	6.29520^{-5}	5.16793^{-5}	0.405500	0.405302
0.6	0.501705	0.502147	2.09836^{-4}	1.53085^{-4}	0.502147	0.501705
0.7	0.605913	0.606778	5.66621^{-4}	3.46026^{-4}	0.606778	0.605913
0.8	0.718799	0.720326	1.30662^{-3}	5.98237^{-4}	0.720326	0.718799
0.9	0.841068	0.843560	2.66190^{-3}	6.99132^{-4}	0.843560	0.841067
1.0	0.973256	0.977074	4.90297^{-3}	6.98894^{-5}	0.977074	0.973251

Table 7: Absolute residual errors | Re | for different values of η at R = 0.1, α = 4.0 in case of **no-slip** and **slip** boundary with γ = 1.0, by the NIM, PM, VIM and ADM.

η	**No-Slip *B***			**Slip *B***		
	NIM&PM	**VIM**	**ADM**	**NIM&PM**	**VIM**	**ADM**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	5.63438^{-15}	5.63438^{-15}	2.31759^{-15}	7.27474^{-13}	7.27474^{-13}	7.22783^{-13}
0.2	7.21923^{-13}	7.21923^{-13}	1.03892^{-12}	9.13051^{-11}	9.13051^{-11}	8.89284^{-11}
0.3	1.14405^{-11}	1.14405^{-11}	6.02401^{-11}	1.50939^{-9}	1.50939^{-9}	1.41859^{-9}
0.4	5.97444^{-11}	5.97444^{-11}	9.70104^{-10}	1.07916^{-8}	1.07916^{-8}	9.60012^{-9}
0.5	2.94232^{-11}	2.94232^{-11}	8.43342^{-9}	4.84153^{-8}	4.84153^{-8}	3.97074^{-8}
0.6	2.46997^{-9}	2.46997^{-9}	5.05292^{-8}	1.60795^{-7}	1.60795^{-7}	1.16926^{-7}
0.7	2.03243^{-8}	2.03243^{-8}	2.35489^{-7}	4.31515^{-7}	4.31515^{-7}	2.60840^{-7}
0.8	1.09632^{-7}	1.09632^{-7}	9.14757^{-7}	9.85311^{-7}	9.85311^{-7}	4.36600^{-7}
0.9	4.37357^{-7}	4.67357^{-7}	3.09276^{-6}	1.97750^{-6}	1.97750^{-6}	4.55002^{-7}
1.0	1.69343^{-6}	1.69343^{-6}	9.36847^{-6}	3.56318^{-6}	3.56318^{-6}	1.19842^{-7}

Table 8: Absolute residual errors | Re | for different values of η at R = 0.3, α = 4.0 in case of **no-slip** and **slip** boundary with γ = 1.0, by the NIM, PM, VIM and ADM.

η	**No-Slip *B***			**Slip *B***		
	NIM&PM	**VIM**	**ADM**	**NIM&PM**	**VIM**	**ADM**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	4.55608^{-13}	4.55608^{-13}	1.86906^{-13}	5.89262^{-11}	5.89262^{-11}	5.85462^{-11}
0.2	5.84710^{-11}	5.84710^{-11}	8.40956^{-11}	7.39595^{-9}	7.39595^{-9}	7.20264^{-9}
0.3	9.26826^{-10}	9.26826^{-10}	4.87359^{-9}	1.22281^{-7}	1.22281^{-7}	1.14927^{-7}
0.4	4.85819^{-9}	4.85819^{-9}	7.84145^{-8}	8.74585^{-7}	8.74585^{-7}	7.78099^{-7}
0.5	1.90307^{-9}	1.90307^{-9}	6.80804^{-7}	3.92681^{-6}	3.92681^{-6}	3.22179^{-6}

0.6	1.93785^{-7}	1.93785^{-7}	4.07194^{-6}	1.30608^{-5}	1.30608^{-5}	9.51000^{-6}
0.7	1.59180^{-6}	1.59180^{-6}	1.89342^{-5}	3.51392^{-5}	3.51392^{-5}	2.13295^{-5}
0.8	8.52601^{-6}	8.52601^{-6}	7.33429^{-5}	8.05617^{-5}	8.05617^{-5}	3.61851^{-5}
0.9	3.59984^{-5}	3.59984^{-5}	2.47124^{-4}	1.62693^{-4}	1.62693^{-4}	3.96350^{-5}
1.0	1.28930^{-4}	1.28930^{-4}	7.45555^{-4}	2.95870^{-4}	2.95870^{-4}	7.42371^{-6}

Table 9: Absolute residual errors | Re | for different values of η at $R = 0.5$, $\alpha = 4.0$ in case of **no-slip** and **slip** boundary with $\gamma = 1.0$, by the NIM, PM, VIM and ADM.

η	**No-Slip *B***			**Slip *B***		
	NIM&PM	**VIM**	**ADM**	**NIM&PM**	**VIM**	**ADM**
0.0	0.000000	0.000000	0.000000	0.000000	0.0000000	0.0000000
0.1	3.51580^{-12}	3.51580^{-12}	1.44218^{-12}	4.54678^{-10}	4.54678^{-10}	4.15746^{-10}
0.2	4.51141^{-10}	4.51141^{-10}	6.48459^{-10}	5.70694^{-8}	5.70694^{-8}	5.55778^{-8}
0.3	7.15258^{-9}	7.15258^{-9}	3.75596^{-8}	9.43682^{-7}	9.43682^{-7}	8.86948^{-7}
0.4	3.76300^{-8}	3.76300^{-8}	6.03788^{-7}	6.75191^{-6}	6.75191^{-6}	6.00761^{-6}
0.5	1.10362^{-8}	1.10362^{-8}	5.23539^{-6}	3.03391^{-6}	3.03391^{-6}	2.49016^{-5}
0.6	1.44776^{-6}	1.44776^{-6}	3.12585^{-5}	1.01056^{-4}	1.01056^{-4}	7.36777^{-5}
0.7	1.18733^{-5}	1.18733^{-5}	1.45021^{-4}	2.72557^{-4}	2.72557^{-4}	1.66113^{-4}
0.8	6.31480^{-5}	6.31480^{-5}	5.60162^{-4}	6.27313^{-4}	6.27313^{-4}	2.85430^{-4}
0.9	2.64053^{-4}	2.64053^{-4}	1.88097^{-3}	1.27436^{-3}	1.27436^{-3}	3.26825^{-4}
1.0	9.34686^{-4}	9.34686^{-4}	5.65177^{-3}	2.33762^{-3}	2.33762^{-3}	3.76189^{-6}

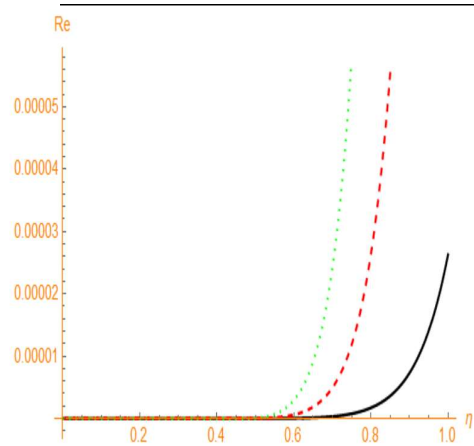

Figure 2(a): Residuals for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0$, by VIM in case of no-slip boundary.

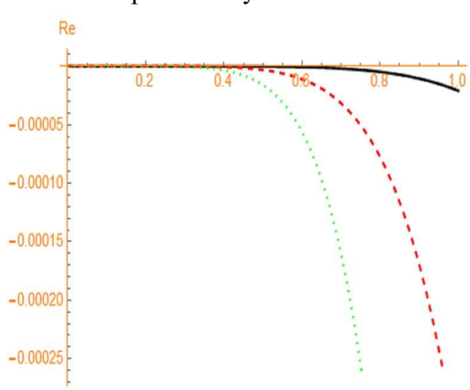

Fig. 2(b): Residuals for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0$, $\gamma = 0.9$, by ADM in case of slip boundary.

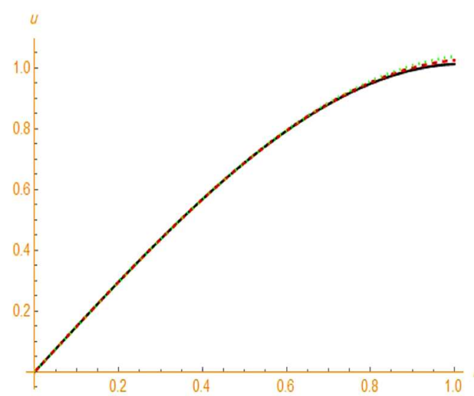

Fig. 3(a): Velocity profiles (Normal velocity) for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0$, by VIM in case of no-slip boundary.

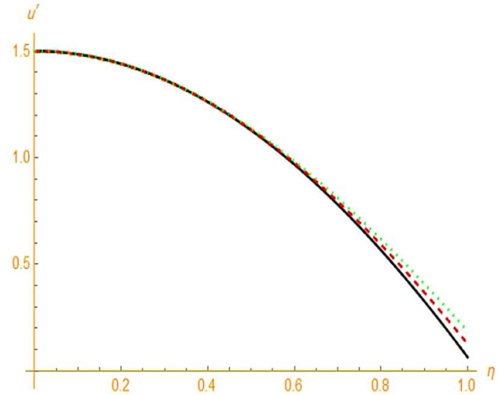

Fig. 3(b): Velocity profiles (Longitudinal Normal velocity) for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0$, by VIM in case of no-slip boundary.

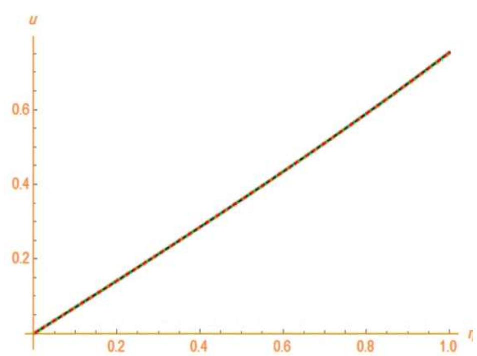

Fig. 4(a): Velocity profiles (Normal velocity) for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0,\ \gamma = 0.9$, by ADM in case of slip boundary.

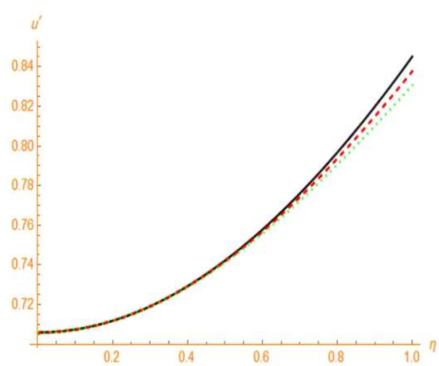

Fig. 4(b): Velocity profiles (Longitudinal velocity) for $R = 0.2$ (continuous curve), 0.4 (dashed curve), 0.6 (dotted curve) at $\alpha = 4.0,\ \gamma = 0.9$, by ADM in case of slip boundary.

7. Conclusion

In this chapter, an analytical solution for an unsteady axisymmetric flow of in compressible, nonconducting Newtonian fluid squeezed between two circular plates in fractional form is obtained using of the VIM and ADM in cases of no-slip and slip boundaries. Analysis of the residual errors confirms that the VIM and ADM are almost identical and efficient schemes. Convergence of the considered methods is confirmed by absolute residual errors for different values of the Reynolds number *R*. The comparisons for the $|\operatorname{Re}|$ for VIM, ADM, and NIM & PM confirmed that the results of the four methods are identical. Therefore, we concluded that the considered methods can be effectively used in various fields of science and engineering as thy give better results in terms of accuracy.

8. Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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